

High-Order Upwind Methods for Second-Order Wave Equations on Overlapping Grids

AMS Seminar Series

Jordan B. Angel

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Outline

Background

Upwind scheme for second-order in time Maxwell's equations and stability on overlapping grids

Optimized upwinding for second-order in time

Upwind scheme for elasticity

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Optimized upwinding for second-order in time

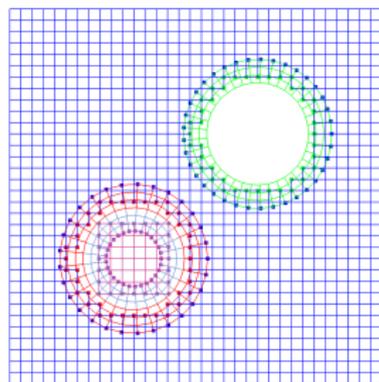
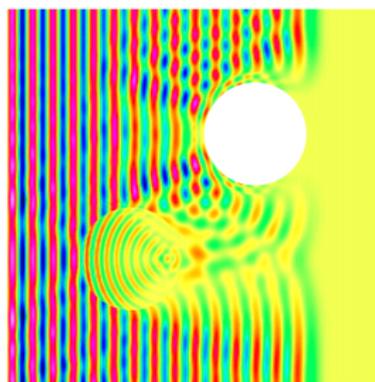
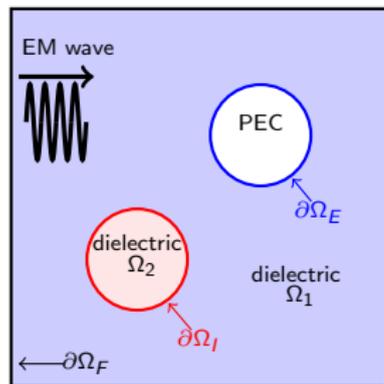
Upwind scheme for elasticity

Maxwell's equations in second-order form

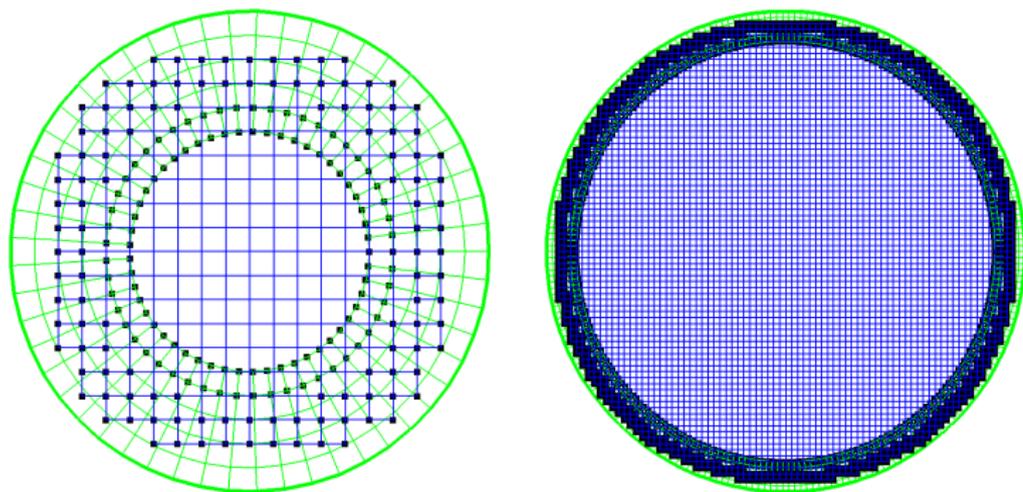
Overlapping structured grids

► Maxwell's Eqn

$$\begin{aligned} \partial_t^2 \mathbf{E} &= c^2 \Delta \mathbf{E}, & \mathbf{x} \in \Omega, \\ \mathbf{n} \times \mathbf{E} &= 0, \quad \nabla \cdot \mathbf{E} = 0, & \mathbf{x} \in \partial\Omega_E, \text{ (PEC BC's)} \\ [\mathbf{n} \times \mathbf{E}]_I &= 0, \quad [\epsilon \mathbf{n} \cdot \mathbf{E}]_I = 0, & \mathbf{x} \in \partial\Omega_I, \text{ (interface conditions)} \end{aligned}$$

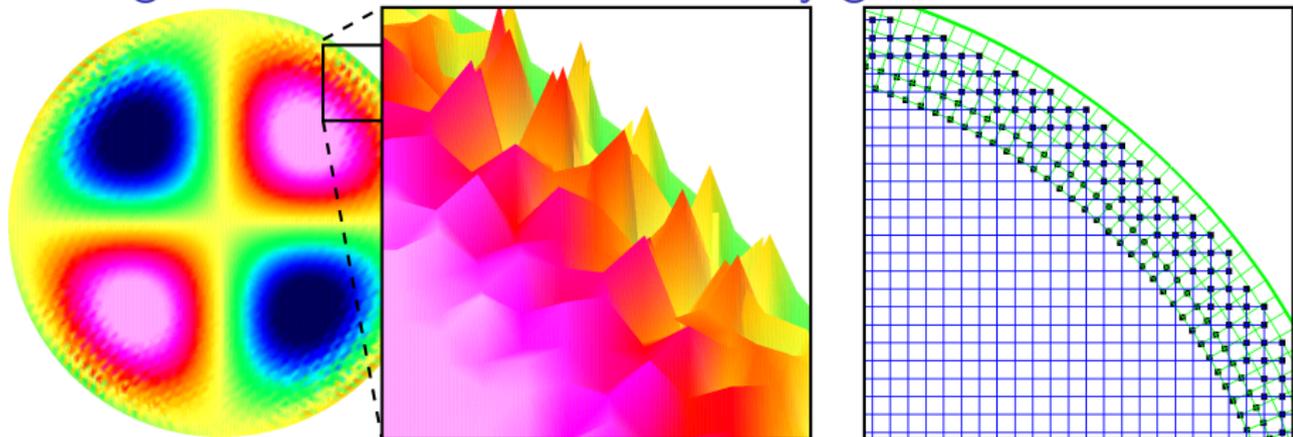


Using thin boundary grids for computational efficiency



- ▶ Solving Maxwell's equations on Cartesian grids drastically more efficient (E.g. factor of 10 for 2d fourth order)
- ▶ For efficiency, want to fix number of radial points
- ▶ Refinement: almost all points on Cartesian but interp. interface \rightarrow physical boundary

Strong instabilities for thin boundary grids



- ▶ It has been shown by Appelö, Banks, Henshaw, and Schwendeman that, in this case, artificial dissipation must scale like inverse of grid spacing \rightarrow pay one order of accuracy to keep stencil compact
- ▶ movie
- ▶ Want something more robust, no manual tuning, no loss of observed accuracy

Upwind for SOWE by Banks and Henshaw (2012)

- ▶ Write $u_{tt} = c^2 u_{xx}$ as first-order system in time

$$\begin{bmatrix} u \\ v \end{bmatrix}_t = \begin{bmatrix} 0 \\ c^2 u_x \end{bmatrix}_x + \begin{bmatrix} v \\ 0 \end{bmatrix} \quad (1a)$$

- ▶ Integrate equation for v in time,

$$v(x, t) = v(x, 0) + c^2 \int_0^t \frac{\partial^2 u}{\partial x^2}(x, \tau) d\tau \quad (1b)$$

- ▶ Take *conservative finite difference* approach and write spatial derivatives as *exact* difference of flux function

$$\frac{\partial^2 u}{\partial x^2} = D_{+x} f(x - h/2, t) \quad (1c)$$

where $D_+ w(x) = (w(x + h) - w(x))/h$

Formally exact conservation form for u and v

- ▶ Integrate v equation over a single time-step Δt two times to get *exact conservation form* for v and u at t^{n+1}

$$v(x, t^{n+1}) = v(x, t^n) + c^2 \Delta t D_+ \mathcal{F}^v(x - h/2, t^n) \quad (2a)$$

$$u(x, t^{n+1}) = u(x, t^n) + \Delta t v(x, t^n) + c^2 \Delta t^2 D_+ \mathcal{F}^u(x - h/2, t^n) \quad (2b)$$

$$\mathcal{F}^v(x, t^n, \Delta t) = \frac{1}{\Delta t} \int_0^{\Delta t} f(x, t^n + \tau) d\tau \quad (2c)$$

$$\mathcal{F}^u(x, t^n, \Delta t) = \frac{1}{\Delta t^2} \int_0^{\Delta t} \int_0^{\tau} f(x, t^n + \tau') d\tau' d\tau \quad (2d)$$

- ▶ Flux $f = \mathcal{S}_x u_x$, \mathcal{S}_x high-order correction operator

Defining the upwind flux function

- ▶ Use d'Alembert solution to solve local Riemann problem, assume u_x is const. at cell-faces but v may jump, embed upwind state u_x^* into $f = \mathcal{S}_x u_x$ to get

$$\hat{f}(x - h/2, t^n + \tau) \stackrel{\text{def}}{=} \mathcal{S}_x \frac{\partial u}{\partial x}(x_{i-1/2}, t^n + \tau) + \mathcal{S}_x \frac{1}{2c} [v^+(x_{i-1/2}, t^n + \tau) - v^-(x_{i-1/2}, t^n + \tau)] \quad (3a)$$

- ▶ Space-time scheme developed by Taylor expanding \hat{f} in space and time and using the PDE to replace time derivatives with space derivatives (Cauchy-Kowaleski procedure)
- ▶ v^+ and v^- gives a left and right biased approximations to v at cell-face. Note: jump in v is formally zero for smooth solutions but nonzero in the discrete approximation and is the source of dissipation

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Conservative curvilinear wave equation

- ▶ Conservation form for second-order wave equation

$$\partial_t^2 u = \frac{1}{J} \partial_\ell \left(J g^{\ell m} \partial_m u \right) = \frac{1}{J} \partial_\ell F^\ell \quad (4a)$$

J is determinant of Jacobian and $g^{\ell m} = \frac{\partial r^\ell}{\partial x^\nu} \frac{\partial r^m}{\partial x^\nu}$

- ▶ Write divergence as exact cell-centered difference of fluxes,

$$\partial_t^2 u = \frac{1}{J} D_{+\ell} f^\ell \left(\mathbf{r} - \frac{\mathbf{h}_\ell}{2}, t \right) \quad (4b)$$

where $D_{+\ell} w(\mathbf{x}, t) = (w(\mathbf{x} + \mathbf{h}_\ell, t) - w(\mathbf{x}, t)) / h_\ell$

- ▶ Flux function (no sum)

$$f^\ell \stackrel{\text{def}}{=} \mathcal{S}^\ell F^\ell \left(\mathbf{r} - \frac{\mathbf{h}_\ell}{2}, t \right) \quad (4c)$$

for linear operator $\mathcal{S}^\ell = \alpha_\nu h_\ell^{2\nu} \partial_\ell^{2\nu}$ where $\alpha_\nu = B_{2\nu}(\frac{1}{2}) / 2\nu!$

Formally exact conservation form for u and v

- ▶ Integrate equation for $\partial_t^2 u$ over a single time-step Δt two times to get *exact conservation form* for $v \stackrel{\text{def}}{=} \partial_t u$ and u at t^{n+1}

$$v(\mathbf{r}, t^{n+1}) = v(\mathbf{r}, t^n) + \frac{\Delta t}{J} D_{+\ell} \mathcal{F}_v^\ell(\mathbf{r} - \mathbf{h}_\ell/2, t^n) \quad (5a)$$

$$\mathcal{F}_v^\ell(\mathbf{r}, t^n) = \frac{1}{\Delta t} \int_0^{\Delta t} f(\mathbf{r}, t^n + \tau) d\tau \quad (5b)$$

$$\begin{aligned} u(\mathbf{r}, t^{n+1}) &= u(\mathbf{r}, t^n) + \Delta t v(\mathbf{r}, t^n) \\ &\quad + \frac{\Delta t^2}{J} D_{+\ell} \mathcal{F}_u^\ell(\mathbf{r} - \mathbf{h}_\ell/2, t^n) \end{aligned} \quad (5c)$$

$$\mathcal{F}_u^\ell(\mathbf{r}, t^n) = \frac{1}{\Delta t^2} \int_0^{\Delta t} \int_0^\tau f(\mathbf{r}, t^n + \tau') d\tau' \quad (5d)$$

- ▶ Space-time scheme: Taylor expand f for small τ , h . Use PDE to replace time derivatives with space derivatives (Cauchy-Kowalevski procedure)

Defining the upwind flux function for curvilinear geometries

- ▶ Define a stress $\sigma^\ell = Jg^{\ell m}\partial_m u$. Solve a generalized Riemann problem using an exact d'Alembert solution. If σ continuous at interface, characteristics give upwind stress (no sum):

$$\sigma^{\ell*} = \sigma^\ell + \frac{Jc\sqrt{g^{\ell\ell}}}{2} [v]_{r^\ell-}^{r^\ell+} \quad (6a)$$

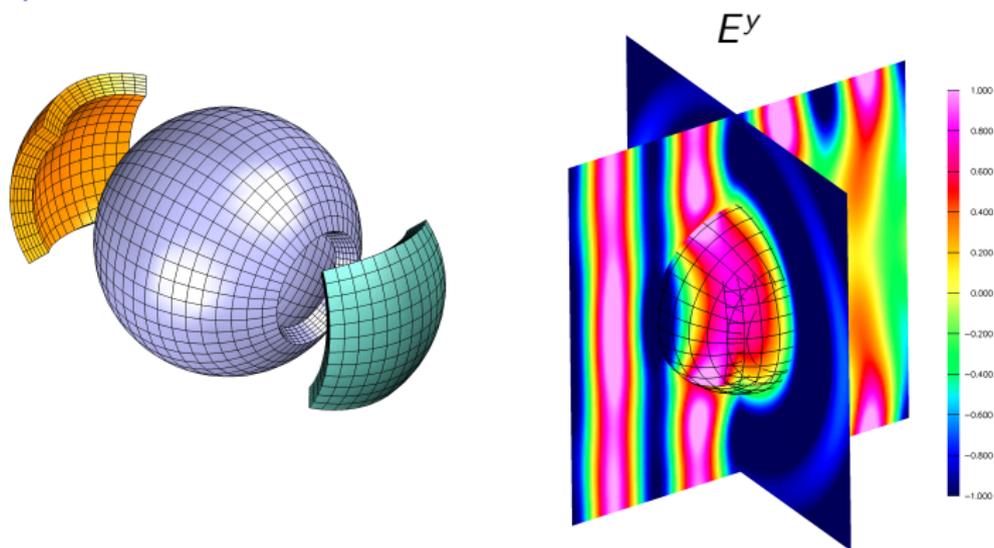
- ▶ Embed upwind state $\sigma^{\ell*}$ into flux function to get

$$\hat{f} \stackrel{\text{def}}{=} f^\ell + \boxed{\mathcal{S}^\ell \frac{Jc\sqrt{g^{\ell\ell}}}{2} (v^+ - v^-)} \quad (6b)$$

- ▶ Note: jump in v is formally zero for smooth solutions, we haven't spoiled exactness

High-order convergence for scattering problems

Dielectric sphere

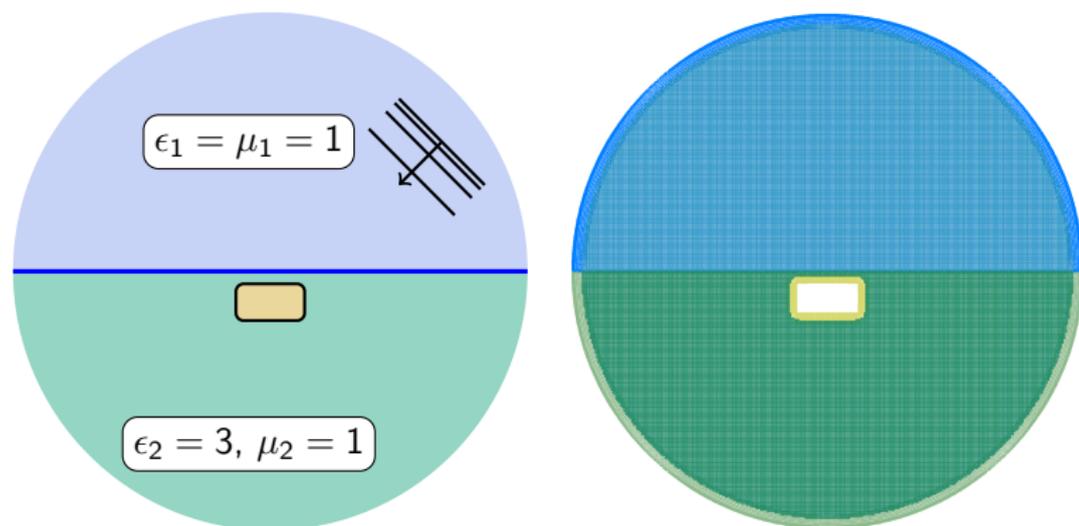


4th Order Max-norm Convergence

h_j	E^x	r	E^y	r	E^z	r	$\partial_t E^x$	r	$\partial_t E^y$	r	$\partial_t E^z$	r
1/20	1.5e-3		5.8e-3		5.6e-4		2.2e-2		5.1e-2		8.8e-3	
1/40	9.4e-5	16.3	1.9e-4	29.6	3.6e-5	15.7	2.1e-3	10.9	3.1e-3	16.6	7.0e-4	12.5
1/80	6.1e-6	15.3	1.2e-5	15.6	2.2e-6	16.0	1.2e-4	16.8	1.8e-4	16.7	4.2e-5	16.8
rate		3.98		4.43		3.98		3.79		4.07		3.86

Upwind scheme accurate and robust for more complex problems

Chirped planewave incident on conducting body

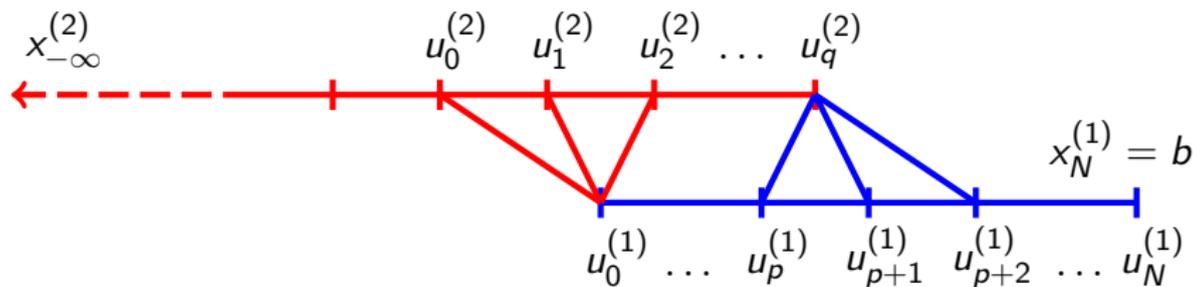


h_j	E_x	r	E_y	r	H_z	r	$\partial_t E_x$	r	$\partial_t E_y$	r	$\partial_t H_z$	r
1/160	1.4e-2		1.6e-2		3.6e-4		5.4e-1		6.1e-1		3.0e-2	
1/320	1.4e-3	10.6	1.5e-3	11.1	2.6e-5	13.8	6.3e-2	8.7	6.5e-2	9.5	1.9e-3	16.0
1/640	1.3e-4	10.6	1.3e-4	11.1	1.9e-6	13.8	7.3e-3	8.7	6.9e-3	9.5	1.2e-4	16.0
rate		3.41		3.47		3.78		3.11		3.24		4.00

Conducting body in dielectric slab

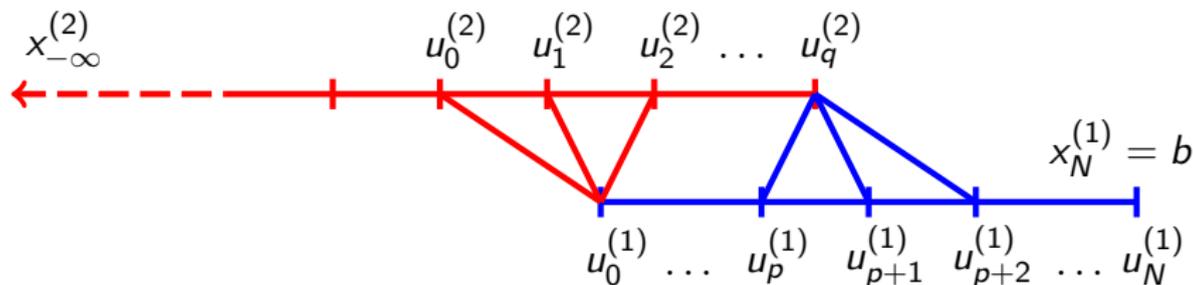
Video

How to determine stability for overlapping grids? Theory of normal modes for IBVP



- ▶ Consider a semi-infinite domain $x \in [\infty, b]$ meshed with two composite grids
- ▶ Stability for us means bounded uniformly in time (strict)
- ▶ Theory of normal modes (GKS) gives an eigenvalue problem for complex number s where $u_j^{(m)}(t) = \kappa_m^j e^{st}$. Unstable modes correspond to $\text{Re}(s) > 0$

Stability issues with overlapping grids



Generalized first-order upwind ($\gamma = 1$) / second-order nondissipative ($\gamma = 0$) semi-discretization

$$\frac{d^2 u_j^{(m)}}{dt^2} = D_+ D_- u_j^{(m)} + \gamma \frac{h_m}{2} D_+ D_- \frac{du_j^{(m)}}{dt}, \quad m = 1, 2 \quad (7a)$$

Boundary and interpolation conditions

$$u_N^{(1)} = 0, \quad \|u^{(2)}\| < \infty, \quad (7b)$$

$$u_0^{(1)} = \sum_{k=0}^r a_k u_k^{(2)}, \quad u_q^{(2)} = \sum_{k=0}^r b_k u_{p+k}^{(1)}. \quad (7c)$$

Formal statement of stability condition

Theorem

Discrete solutions to the one-dimensional overlapping grid problem defined by equations (7a)–(7c) are stable (no exponential growth in time) provided there are no solutions $(s, \kappa_1(s), \kappa_2(s))$ to the system of polynomial equations

$$G(s, \kappa_1, \kappa_2) = 0, \quad (8a)$$

$$\left(1 + \gamma \frac{z_1}{2}\right) \kappa_1^2 - (z_1^2 + \gamma z_1 + 2) \kappa_1 + \left(1 + \gamma \frac{z_1}{2}\right) = 0, \quad (8b)$$

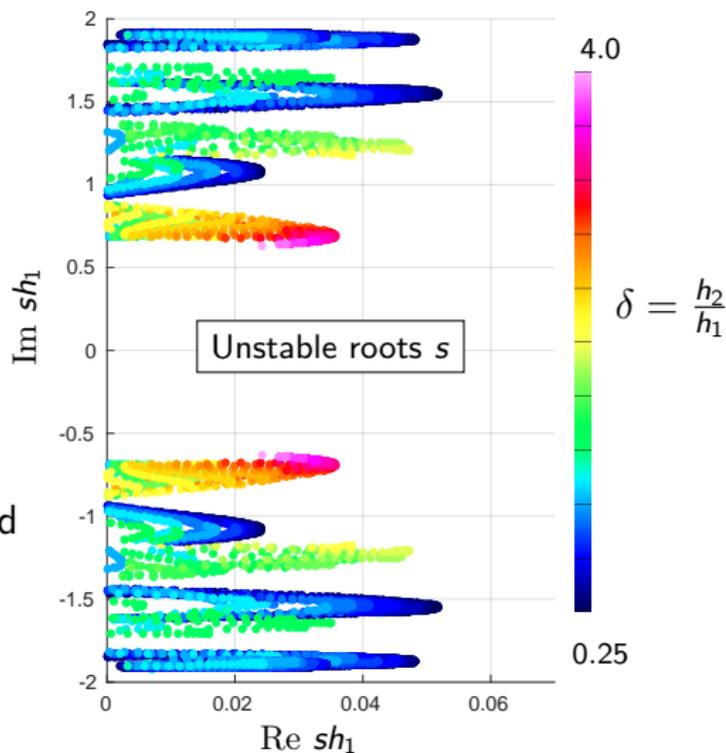
$$\left(1 + \gamma \frac{z_2}{2}\right) \kappa_2^2 - (z_2^2 + \gamma z_2 + 2) \kappa_2 + \left(1 + \gamma \frac{z_2}{2}\right) = 0 \quad (8c)$$

that satisfy $|\kappa_m| < 1$ for $m = 1, 2$, and $\text{Re}(s) > 0$, where $z_m = sh_m$ and

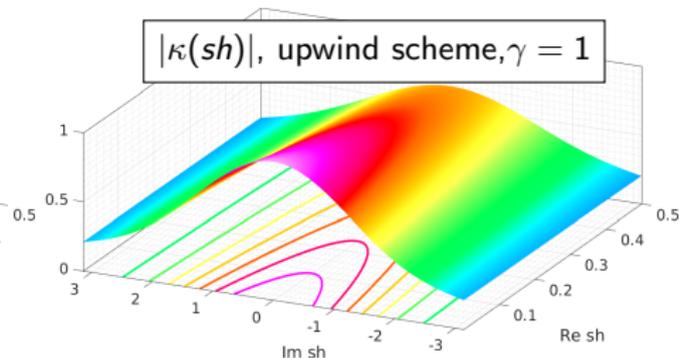
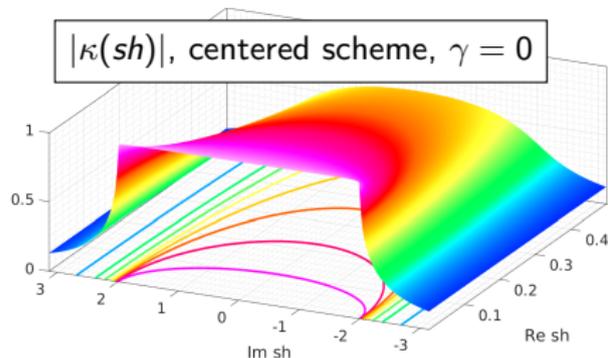
$$G(s, \kappa_1, \kappa_2) \stackrel{\text{def}}{=} 1 - \kappa_1^{2N} - \left(\sum_{k=0}^r a_k \kappa_2^{r-k}\right) \left(\sum_{k=0}^r b_k \left(\kappa_1^k - \kappa_1^{2N-2p-k}\right)\right) \kappa_1^p \kappa_2^{q-r} = 0$$

Solving polynomial system for many grid configurations with $\gamma = 0$

- ▶ Very easy to find unstable modes for nondissipative scheme
- ▶ Worst cases mostly at integral values of δ
- ▶ No instability for low-frequency, well-resolved modes



Strong decay in $|\kappa_m|$ parallel to $\text{Im}(sh_m)$



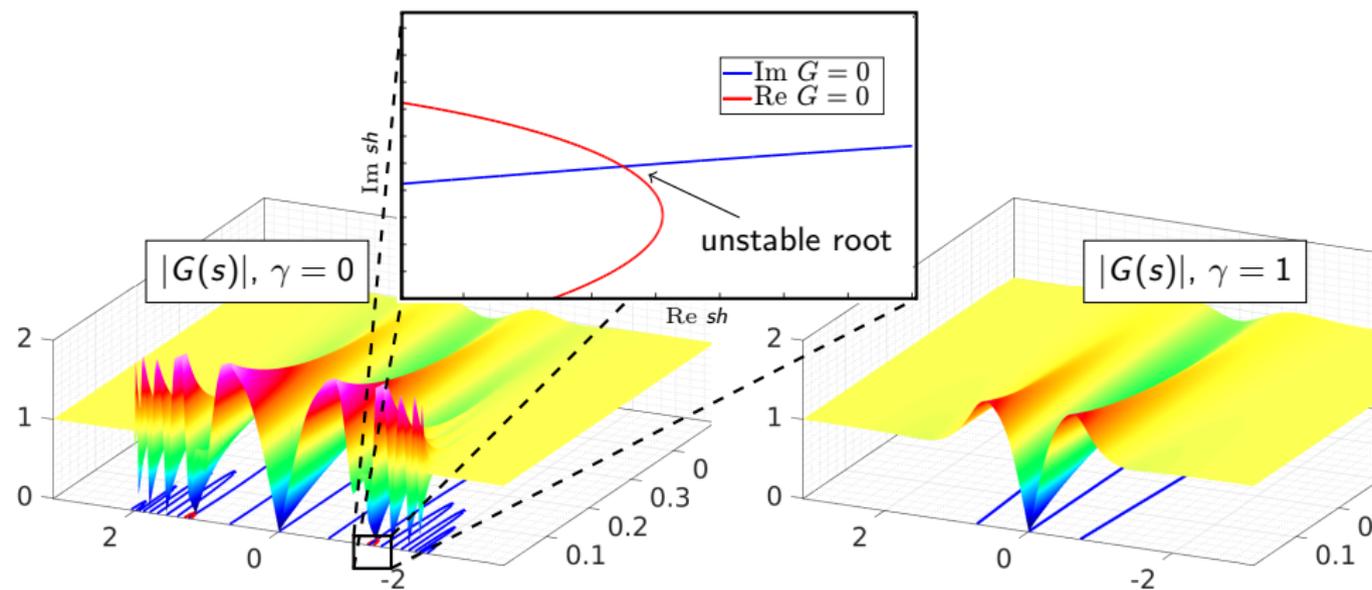
- For small $|sh_m|$,

$$|\kappa_m| \sim \exp\left(-\xi_m + \frac{\gamma}{4}(\xi_m^2 - \eta_m^2)\right)$$

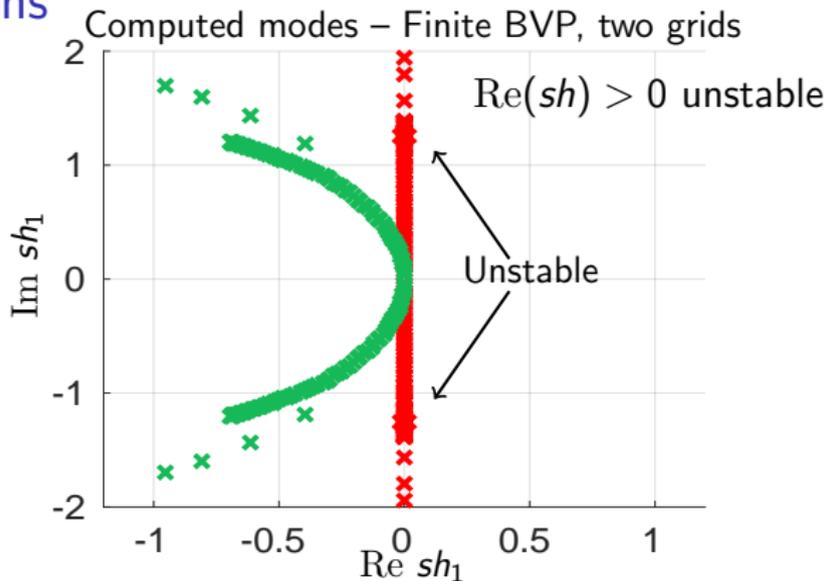
then with $sh_m = \xi_m + i\eta_m$, The additional damping is represented by the term $-\frac{\gamma}{4}\eta_m^2$.

- Analysis shows unstable modes must have $|\kappa_m| \approx 1$

Determinant $G(s)$ well-behaved for upwind scheme

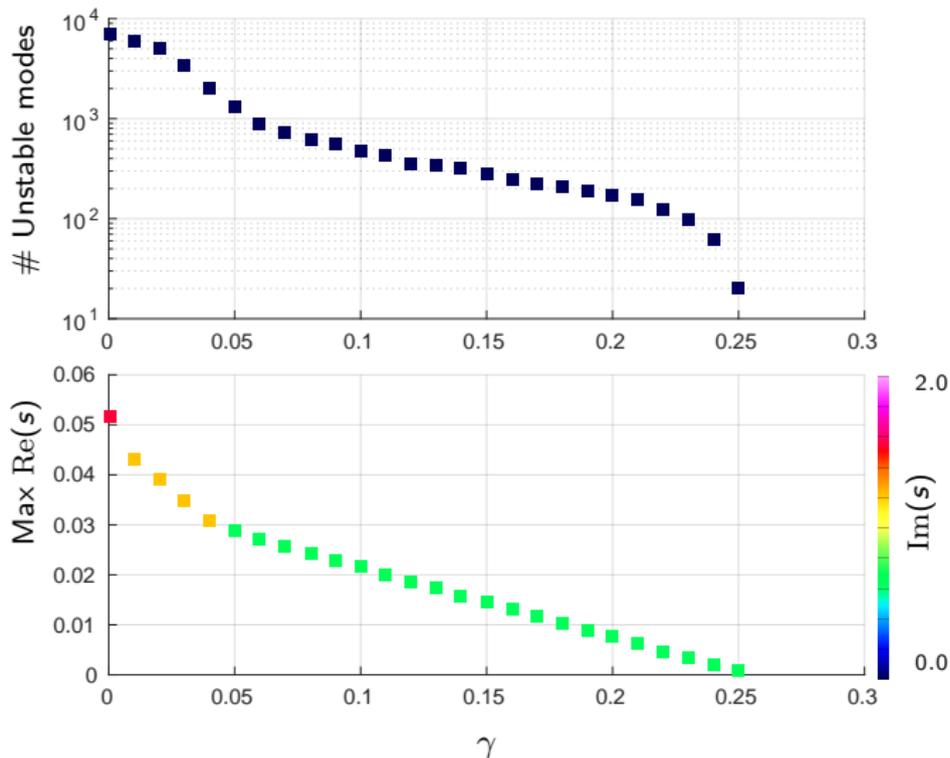


Eigenvalues for overlapping grid problem stable to perturbations



- ▶ With no dissipation, any small perturbation of eigenvalues gives instability
- ▶ Upwind scheme is robust with respect to these perturbations since all eigenvalues far into region of stability

Solving polynomial system for $\gamma = j/100, j = 1, \dots, 100$



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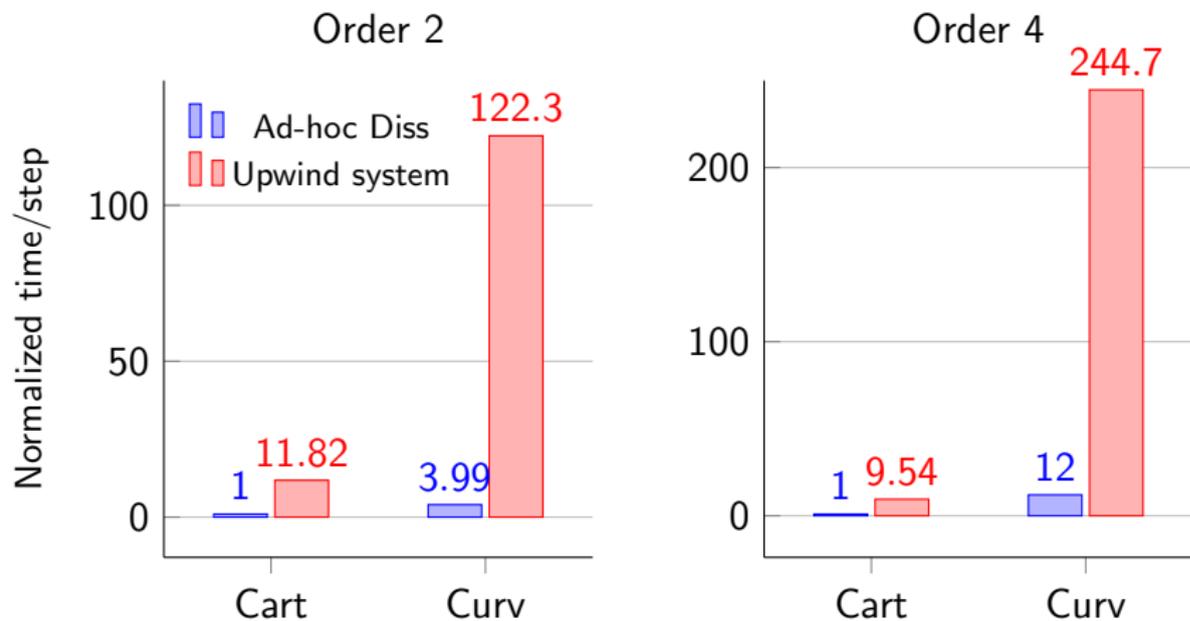
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Optimized upwinding for second-order in time

Upwind scheme for elasticity

Implementation of upwind scheme not optimal

Time per step, 2D, $\approx 1\text{M}$ grid points



- ▶ Ratio of normalization factors $\approx \frac{9}{5}$

Recalling the upwind flux for curvilinear coordinates

First optimization

- ▶ Upwind flux is given by

$$\hat{f} \stackrel{\text{def}}{=} f^\ell + \mathcal{S}^\ell \frac{Jc\sqrt{g^{ll}}}{2} (v^+ - v^-) \quad (10a)$$

- ▶ Jump term for v is on the order of truncation error
- ▶ v can be replaced by a low-order difference in time of u
- ▶ Resulting scheme only involves u , no auxiliary variable needed

Second-order in time scheme

Taylor time-stepping and Cauchy-Kowalevski procedure

- ▶ We are solving $u_{tt} = \mathcal{L}u$. Observe that

$$D_{+t}D_{-t}u^n = u_{tt} + \frac{\Delta t^2}{12}u_{tttt} + \frac{\Delta t^4}{360}u_{tttttt} + \mathcal{O}(\Delta t^6) \quad (11a)$$

$$= \mathcal{L}u + \frac{\Delta t^2}{12}\mathcal{L}^2u + \frac{\Delta t^4}{360}\mathcal{L}^3u + \mathcal{O}(\Delta t^6) \quad (11b)$$

- ▶ Approximating \mathcal{L}^m to appropriate order to get both space and time accuracy. E.g.

$$D_{+t}D_{-t}u = \mathcal{L}_{(4h)}u + \frac{\Delta t^2}{12}\mathcal{L}_{(2h)}^2u + \mathcal{O}(\Delta t^4) \quad (11c)$$

is a fourth-order scheme in space and time

d -Dimensional formally exact conservation form

$$\mathcal{L}u = \frac{1}{J} \left\{ \sum_{\ell=1}^d D_{+r_\ell} \left(\mathcal{S}_{r_\ell} \left(J \sum_{m=1}^d g^{\ell m} \frac{\partial u}{\partial r_m} \right) \right) + \sum_{\ell=1}^d D_{+r_\ell} \mathcal{S}_{r_\ell} \left(\frac{J \sqrt{g^{\ell\ell}}}{2} (\dot{u}^{+r_\ell} - \dot{u}^{-r_\ell}) \right) \right\} \quad (12)$$

- ▶ First term, double sum, represents conservative centered nondissipative wave operator
- ▶ Second term, single sum (!), Diss. term

Only do upwinding on leading order operator

Second optimization

- ▶ Dissipation term isolated from wave operator, $\mathcal{L}u = Lu + M[\dot{u}]$
- ▶ Only do upwinding on leading operator,

$$D_{+t}D_{-t}u^n = Lu + M[\dot{u}] + \frac{\Delta t^2}{12}L^2u + \frac{\Delta t^4}{360}L^3u + \mathcal{O}(\Delta t^6)$$

- ▶ For curvilinear, L is double sum, M single sum, only nonzero in coord. directions
- ▶ Keeps dissipation stencil thin, computationally cheap

How to approximate \dot{u} ?

- ▶ Already using

$$u_{tt} \approx \frac{u^{n+1} - 2u^n - u^{n-1}}{\Delta t^2} \quad (14)$$

- ▶ One natural choice: $\dot{u} \approx (u^n - u^{n-1})/\Delta t$
 - ▶ Like artificial diss, wider stencil, coefficient given
 - ▶ Time step restriction, e.g. fourth-order 1D: $c\Delta t/h_x < 0.71\dots$
- ▶ Could avoid time step restriction with implicit update $\dot{u} \approx (u^{n+1} - u^{n-1})/(2\Delta t)$
 - ▶ CFL restriction, 1D: $c\Delta t/h_x \leq 1$, same as wave operator
 - ▶ Requires inverting a matrix every time step
- ▶ Instead, take a predictor-corrector approach

General construction of predictor-corrector scheme

- ▶ Predict solution using nondissipative wave operator,

$$u^p = 2u^n - u^{n-1} + \Delta t^2 Lu + \frac{\Delta t^4}{12} L^2 u + \frac{\Delta t^6}{360} L^3 u + \mathcal{O}(\Delta t^8) \quad (15a)$$

- ▶ Use predicted state to approximate \dot{u} ,

$$\dot{u} \approx (u^p - u^{n-1}) / (2\Delta t) \quad (15b)$$

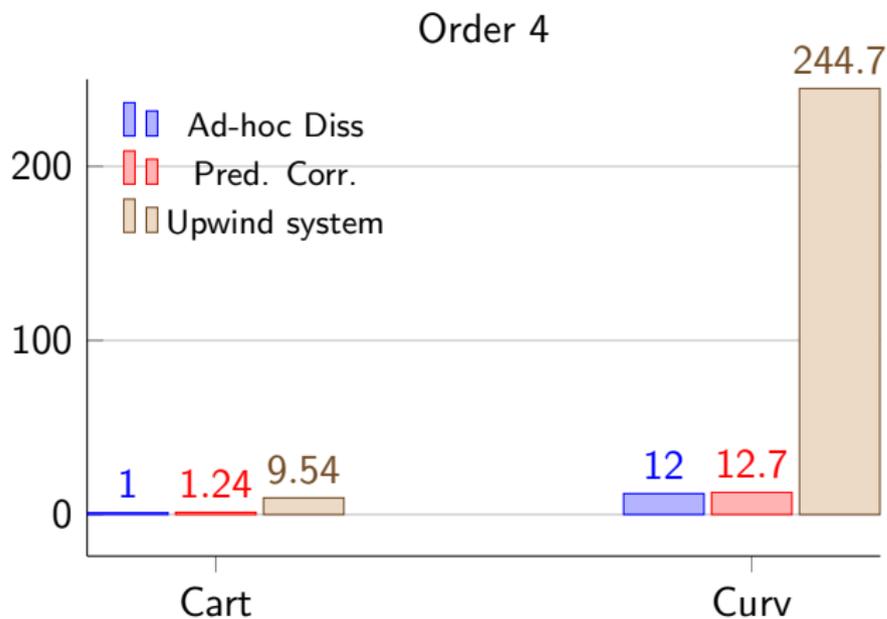
- ▶ Finally, correct by adding upwind dissipation

$$u^{n+1} = u^p + \Delta t^2 M[\dot{u}] \quad (15c)$$

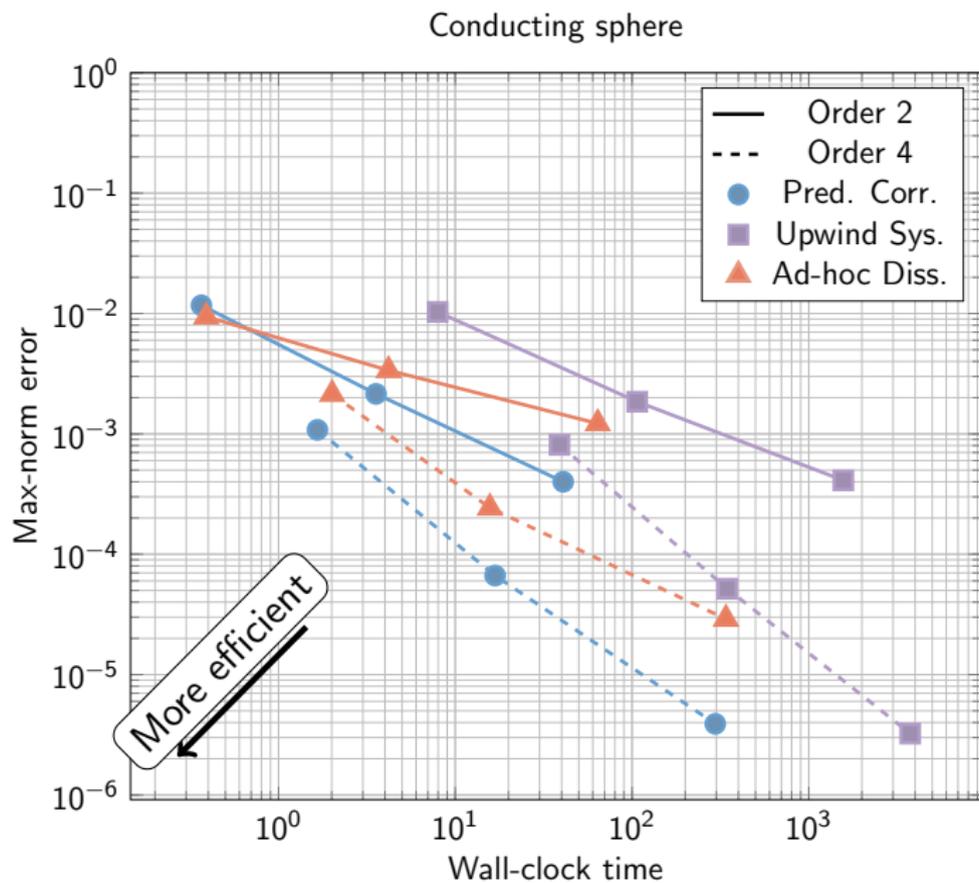
- ▶ CFL restriction 1D: $c\Delta t/h_x \leq 1$, same as only using wave-operator, no CFL hit

New scheme provides significant speed-up

Time per step, 2D, $\approx 1M$ grid points



Predictor-corrector scheme is fast and accurate



New scheme enables simulating designer metamaterials for novel optics

- ▶ Engineered materials with bulk material properties not found in nature
- ▶ Sub wavelength diffraction, optical computing, cloaking

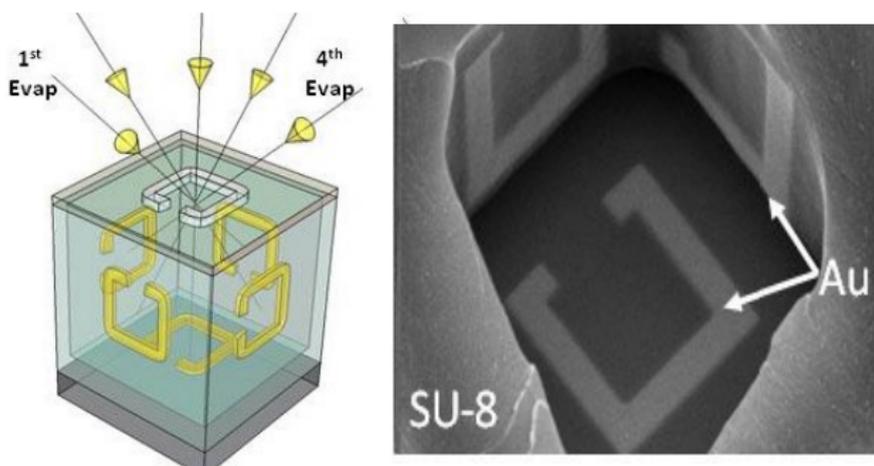
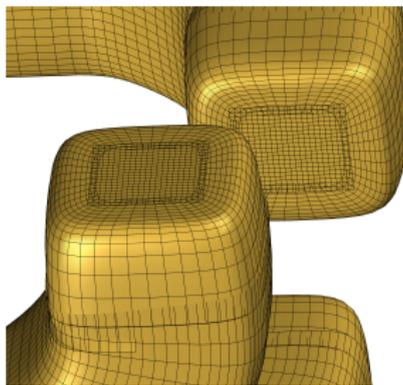
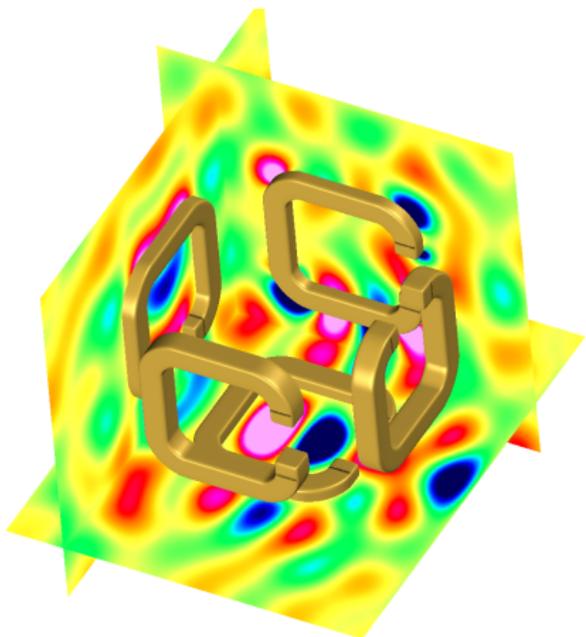


Figure : images were accessed from <http://spie.org/newsroom/3174-3d-metamaterials-for-thermal-ir-applications>

Need fast, accurate solutions of Maxwell's equations

Optimized scheme makes larger simulations feasible on modest hardware

- ▶ Ad-hoc diss. insufficient for stability, original upwind scheme very expensive
- ▶ $\approx 12\text{M}$ grid points, practical for laptop
- ▶ video



Overall conclusions

- ▶ Efficient stabilizing upwind dissipation, easy to incorporate into existing codes
- ▶ For fixed error tolerance, pred-corr. upwind scheme much more efficient
- ▶ Preliminary work for dispersive Maxwell's equations
- ▶ New approach enables fast, accurate, stable calculations for wave equation on overlapping grids, essentially same cost as Cartesian domain

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Upwind scheme for elasticity

Linear elasticity – Nontrivially coupled system

- ▶ Elastic wave equations for horizontal and vertical displacement u and v ,

$$\rho u_{tt} = ((2\mu + \lambda)u_x + \lambda v_y)_x + (\mu v_x + \mu u_y)_y \quad (16a)$$

$$\rho v_{tt} = (\mu v_x + \mu u_y)_x + (\lambda u_x + (2\mu + \lambda)v_y)_y \quad (16b)$$

where $\rho = \rho(x, y)$, $\mu = \mu(x, y)$, $\lambda = \lambda(x, y)$.

- ▶ Write as conservative difference of fluxes by introducing stresses σ^{xx} , σ^{xy} , σ^{yy}
- ▶ Reformulate as first-order system to do characteristic analysis,

$$\mathbf{w}_t + A\mathbf{w}_x + B\mathbf{w}_y = 0 \quad (16c)$$

Finding upwind state by computing flux across cell-face

- ▶ Rewrite system as a derivative in a direction \mathbf{n}

$$\mathbf{w}_t + \hat{A}\mathbf{w}_n = 0, \quad \hat{A} = n_1A + n_2B \quad (17a)$$

- ▶ Diagonalize to find characteristic variables and characteristic speeds $\{\pm c_p, \pm c_s, 0\}$ where

$$c_p = \sqrt{\frac{2\mu + \lambda}{\rho}}, \quad c_s = \sqrt{\frac{\mu}{\rho}} \quad (17b)$$

- ▶ Setup a Riemann problem with left and right states across lines of constant x or y . Assume σ 's are continuous. Solve to see upwind fluxes

$$\sigma^{xx} = \sigma^{xx} + \frac{\rho c_p}{2} [u_t]^x \quad (18a)$$

$$\sigma^{xy} = \sigma^{xy} + \frac{\rho c_s}{2} [v_t]^x \quad (18b)$$

$$\sigma^{yx} = \sigma^{yx} + \frac{\rho c_s}{2} [u_t]^y \quad (18c)$$

$$\sigma^{yy} = \sigma^{yy} + \frac{\rho c_p}{2} [v_t]^y \quad (18d)$$

- ▶ Square brackets indicate jump, superscript indicates w.r.t to which variable
- ▶ Have fourth-order conservative finite difference code based these fluxes

High-order correction operator \mathcal{S}_x

- ▶ Set

$$f(x, t) = \mathcal{S}_x \frac{\partial u}{\partial x}(x, t) \quad (19a)$$

such that

$$\frac{\partial^2 w}{\partial x^2}(x - h/2, t) = D_{+x} \left(\mathcal{S}_x \frac{\partial u}{\partial x}(x - h/2, t) \right) \quad (19b)$$

- ▶ \mathcal{S}_x is a generalized shift operator,

$$\mathcal{S}_x = \sum_{\nu=0}^{\infty} \alpha_{\nu} h^{2\nu} \frac{\partial^{2\nu}}{\partial x^{2\nu}} \quad (19c)$$

where $\alpha_{\nu} = B_{2\nu}(1/2)/2\nu!$

- ▶ Expanding a few terms,

$$\mathcal{S}_x u(x, t) = u(x, t) - \frac{h^2}{24} \frac{\partial^2 u}{\partial x^2}(x, t) + \frac{7h^4}{5760} \frac{\partial^4 u}{\partial x^4}(x, t) + \dots \quad (19d)$$

Stability for IBVP (GKS Theory)

Laplace transform in t with dual variable s (assuming homogeneous initial data)

$$s^2 \tilde{u}_j^{(m)} = \frac{1}{h_m^2} \left(\tilde{u}_{j+1}^{(m)} - 2\tilde{u}_j^{(m)} + \tilde{u}_{j-1}^{(m)} \right) + \gamma \frac{s}{2h_m} \left(\tilde{u}_{j+1}^{(m)} - 2\tilde{u}_j^{(m)} + \tilde{u}_{j-1}^{(m)} \right), \quad (20a)$$

with transformed boundary and interpolation conditions

$$\tilde{u}_N^{(1)} = 0, \quad \|\tilde{u}^{(2)}\| < \infty, \quad (20b)$$

$$\tilde{u}_0^{(1)} = \sum_{k=0}^r a_k \tilde{u}_k^{(2)}, \quad \tilde{u}_q^{(2)} = \sum_{k=0}^r b_k \tilde{u}_{p+k}^{(2)}. \quad (20c)$$

Let $\tilde{u}_j^{(m)} = \kappa_m^j$ and substitute into (20a) gives resolvent equation,

$$\left(1 + \gamma \frac{z_m}{2} \right) \kappa_m^2 - (z_m^2 + \gamma z_m + 2) \kappa_m + \left(1 + \gamma \frac{z_m}{2} \right) = 0, \quad (20d)$$

where $z_m = sh_m$.

Make GKS assumption that $\text{Re}(s) > 0$ and look for solutions

$$\kappa_m = \begin{cases} \kappa_m^- & \text{for } \eta_m^2 \leq \frac{\xi_m(2\xi_m^2+3\gamma\xi_m+4)}{2\xi_m+\gamma}, \\ \kappa_m^+ & \text{for } \eta_m^2 > \frac{\xi_m(2\xi_m^2+3\gamma\xi_m+4)}{2\xi_m+\gamma}, \end{cases} \quad (21a)$$

for $sh_m = \xi_m + i\eta_m$ so that $|\kappa_m| < 1$ for $\text{Re}(z_m) > 0$.

$$\tilde{u}_j^{(1)} = A(s)(\kappa_1^j - \kappa_1^{2N-j}) \quad j = 0, 1, \dots, N-1, N \quad (21b)$$

$$\tilde{u}_j^{(2)} = B(s)\kappa_2^{q-j} \quad j = \dots, q-1, q \quad (21c)$$

Interpolation

$$A(1 - \kappa_1^{2N}) = B \sum_{k=0}^r a_k \kappa_2^{q-k} \quad (21d)$$

$$B = A \sum_{k=0}^r b_k \left(\kappa_1^{p+k} - \kappa_1^{2N-(p+k)} \right), \quad (21e)$$

Algebraic conditions for stability

Leads to a homogeneous linear system for A and B ,

$$\begin{bmatrix} \kappa_1^{2N} - 1 & \sum_{k=0}^r a_k \kappa_2^{q-k} \\ \sum_{k=0}^r b_k \left(\kappa_1^{p+k} - \kappa_1^{2N-(p+k)} \right) & -1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (22a)$$

For nontrivial solutions to (22a) to exist, the determinant of the matrix in (22a) must vanish,

$$\begin{aligned} G(s, \kappa_1, \kappa_2) &\stackrel{\text{def}}{=} 1 - \kappa_1^{2N} \\ &- \left(\sum_{k=0}^r a_k \kappa_2^{r-k} \right) \left(\sum_{k=0}^r b_k \left(\kappa_1^k - \kappa_1^{2N-2p-k} \right) \right) \kappa_1^p \kappa_2^{q-r} = 0 \end{aligned} \quad (22b)$$

When can we prove there are no unstable modes?

- ▶ The impractical case: If we allow the number of points on the boundary fitted grid to grow large as we refine the grid and keep the overlap fixed, we can show that there are no solutions to the constrained polynomial system.
- ▶ But this is precisely the refinement process we wanted to avoid by fixing the number of points on the boundary fitted grid
- ▶ Practical case: Keeping the number of points on boundary fitted grid fixed, we are unable to prove that there are no solutions to the polynomial system with $|\kappa_m| < 1$ and $\text{Re}(s) > 0$.
- ▶ Instead of a formal proof, we present some analytical results that suggest why the upwind scheme remains stable and show stability by conducting a careful numerical search for unstable modes

Searching a parameter space for unstable modes

- ▶ Provide convincing evidence of stability without a formal proof
- ▶ Specify grid configuration and solve for all roots of polynomial system (8a)– (8b)
- ▶ Roots can be computed a variety of ways (Gröebner basis, homotopy continuation, argument principle...). Results shown for homotopy method (hybrid analytic-numerical), many results confirmed with Gröebner basis (exact)
- ▶ Overlapping grid characterized by $\delta = \Delta x^{(2)} / \Delta x^{(1)}$ and off-set $x_0^{(2)}$. Define reasonable bounds for δ
- ▶ Valid grids satisfy a centered interpolation condition
- ▶ Choose number of points, M_δ and M_x in search space for δ and $x_0^{(2)}$

Deriving upwind state from characteristic analysis

- ▶ Convert $u_{tt} = c^2 u_{xx}$ to first-order system with $v = \rho u_t$ and $\sigma = c^2 u_x$
- ▶ Diagonalize into characteristic variable and solve a Riemann problem for left and right states (v_L, σ_L) and (v_R, σ_R)
- ▶ Assume σ continuous ($\sigma = \sigma_R = \sigma_L$) and solve for positive time to get upwind state

$$\sigma^* = \sigma + \frac{c\rho}{2} [v_R - v_L] \quad (23)$$

- ▶ Impedance times a jump in velocity add dissipation

Stability for many boundary points

A theorem

Lemma

There are no unstable solutions to the one-dimensional overlapping grid problem when

$$|\kappa_1|^{2N} + C_r^2(1 + |\kappa_1|^{2N-2p-r})|\kappa_1|^p |\kappa_2|^{q-r} < 1 \quad (24)$$

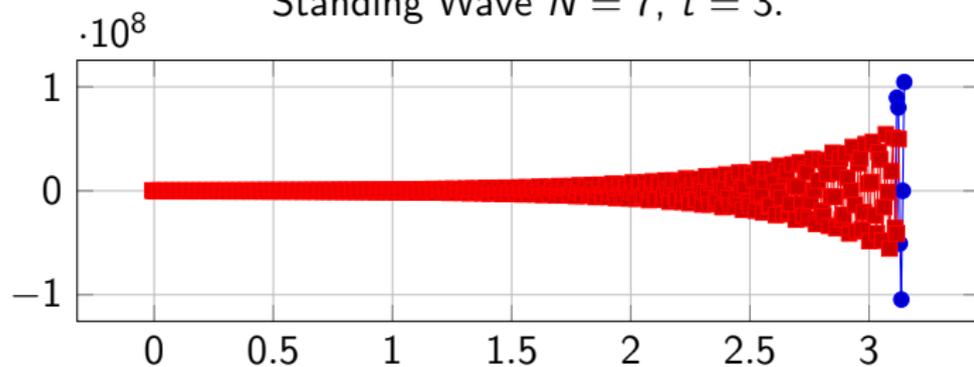
where $C_r = \max \{ \sum_{k=0}^r |a_k|, \sum_{k=0}^r |b_k| \}$.

Proof.

Triangle inequality on (8a) Using $|\kappa_m| < 1$, $r > 0$, $p \geq 0$, $q - r \geq 0$, $N > 0$ and $2N - 2p - r \geq 0$ gives $|G(s, \kappa_1, \kappa_2)| > 0$ when $|\kappa_1|^{2N} + C_r^2(1 + |\kappa_1|^{2N-2p-r})|\kappa_1|^p |\kappa_2|^{q-r} < 1$ □

Computation

Standing Wave $N = 7, t = 3.$



Standing Wave $N = 43, t = 3.$

