

**Taylor Table To Derive a Generalized 4^{th} to 6^{th}
Order Compact Pade Scheme**

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Taylor Tables

1. The generalized form of the equation is given by

$$\left(\frac{\partial u}{\partial x}\right)_{j-1} + \alpha \left(\frac{\partial u}{\partial x}\right)_j + \left(\frac{\partial u}{\partial x}\right)_{j+1} - \frac{A}{2\Delta x} (-u_{j-1} + u_{j+1}) - \frac{B}{4\Delta x} (-u_{j-2} + u_{j+2}) = \text{err}_t$$

1. The equation is written on terms of the single free coefficients α, A, B which must be determined using the Taylor Table approach as outlined below.
2. One goal is to find the value of α, A, B which results in a 6th Order Scheme
3. We can also define a class of 4th schemes where α is a free parameter and A, B are functions of α

Taylor Table

	u_j	$\left(\frac{\Delta x \cdot \partial u}{\partial x}\right)_j$	$\left(\frac{\Delta x^2 \cdot \partial^2 u}{\partial x^2}\right)_j$	$\left(\frac{\Delta x^3 \cdot \partial^3 u}{\partial x^3}\right)_j$	$\left(\frac{\Delta x^4 \cdot \partial^4 u}{\partial x^4}\right)_j$	$\left(\frac{\Delta x^5 \cdot \partial^5 u}{\partial x^5}\right)_j$	$\left(\frac{\Delta x^6 \cdot \partial^6 u}{\partial x^6}\right)_j$	$\left(\frac{\Delta x^7 \cdot \partial^7 u}{\partial x^7}\right)_j$
$\Delta x \cdot \left(\frac{\partial u}{\partial x}\right)_{j-1}$		1	$(-1) \frac{1}{1!}$	$(-1)^2 \frac{1}{2!}$	$(-1)^3 \frac{1}{3!}$	$(-1)^4 \frac{1}{4!}$	$(-1)^5 \frac{1}{5!}$	$(-1)^6 \frac{1}{6!}$
$\Delta x \cdot \alpha \left(\frac{\partial u}{\partial x}\right)_j$		α						
$\Delta x \cdot \left(\frac{\partial u}{\partial x}\right)_{j+1}$		1	$(1) \frac{1}{1!}$	$(1)^2 \frac{1}{2!}$	$(1)^3 \frac{1}{3!}$	$(1)^4 \frac{1}{4!}$	$(1)^5 \frac{1}{5!}$	$(1)^6 \frac{1}{6!}$
$\frac{A}{2} u_{j-1}$	$\frac{A}{2}$	$\frac{A}{2} (-1) \frac{1}{1!}$	$\frac{A}{2} (-1)^2 \frac{1}{2!}$	$\frac{A}{2} (-1)^3 \frac{1}{3!}$	$\frac{A}{2} (-1)^4 \frac{1}{4!}$	$\frac{A}{2} (-1)^5 \frac{1}{5!}$	$\frac{A}{2} (-1)^6 \frac{1}{6!}$	$\frac{A}{2} (-1)^7 \frac{1}{7!}$
$-\frac{A}{2} u_{j+1}$	$-\frac{A}{2}$	$-\frac{A}{2} (1) \frac{1}{1!}$	$-\frac{A}{2} (1)^2 \frac{1}{2!}$	$-\frac{A}{2} (1)^3 \frac{1}{3!}$	$-\frac{A}{2} (1)^4 \frac{1}{4!}$	$-\frac{A}{2} (1)^5 \frac{1}{5!}$	$-\frac{A}{2} (1)^6 \frac{1}{6!}$	$-\frac{A}{2} (1)^7 \frac{1}{7!}$
$\frac{B}{4} u_{j-2}$	$\frac{B}{4}$	$\frac{B}{4} (-2) \frac{1}{1!}$	$\frac{B}{4} (-2)^2 \frac{1}{2!}$	$\frac{B}{4} (-2)^3 \frac{1}{3!}$	$\frac{B}{4} (-2)^4 \frac{1}{4!}$	$\frac{B}{4} (-2)^5 \frac{1}{5!}$	$\frac{B}{4} (-2)^6 \frac{1}{6!}$	$\frac{B}{4} (-2)^7 \frac{1}{7!}$
$-\frac{B}{4} u_{j+2}$	$-\frac{B}{4}$	$-\frac{B}{4} (2) \frac{1}{1!}$	$-\frac{B}{4} (2)^2 \frac{1}{2!}$	$-\frac{B}{4} (2)^3 \frac{1}{3!}$	$-\frac{B}{4} (2)^4 \frac{1}{4!}$	$-\frac{B}{4} (2)^5 \frac{1}{5!}$	$-\frac{B}{4} (2)^6 \frac{1}{6!}$	$-\frac{B}{4} (2)^7 \frac{1}{7!}$
=								
$\Delta x \cdot er_t$	0	0	0	0	0	?	0	?

Coefficient Equations

1. For Consistency and at least 4th Order Accuracy, the first five columns are set to zero.
2. Note because of the skew symmetry of the original equation the odd numbers columns sum exact to 0
3. For 6th Order Accuracy we need

$$\alpha + 2 - A - B = 0, \quad 1 - \frac{A}{6} - \frac{2B}{3} = 0, \quad 2 - \frac{A}{5} - \frac{16B}{5} = 0 \quad (1)$$

1. Solving we have $\alpha = 3$, $A = \frac{14}{3}$, and $B = \frac{1}{3}$
2. Then $er_t = -\frac{1}{180} \Delta x^6 \left(\frac{\partial^7 u}{\partial x^7} \right)_j$

Generalized Scheme

1. Instead of requiring a 6th order scheme relax the conditions to allow the sixth column to be non-zero and find A, B as a function of α
2. Solving the first two relations for A, B we have $A = \frac{4+2\alpha}{3}$ and $B = \frac{4-\alpha}{3}$
3. For $\alpha = 3$: the above 5 point 6th Order Scheme
4. For $\alpha = 4$: the 3 point 4th Order Scheme
5. For $\alpha \neq 3$: a class of 4th Order Schemes different characteristics.
6. In general,
$$er_t = \Delta x^4 \frac{1}{10} \left(1 - \frac{\alpha}{3}\right) \left(\frac{\partial^5 u}{\partial x^5}\right)_j + \Delta x^6 \frac{1}{1260} (8 - 5\alpha) \left(\frac{\partial^7 u}{\partial x^7}\right)_j$$
7. See Note on Modified Wave Number for General Pade Schemes