

# Chapter 4

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## PDE to ODE Matrix

1. In general, cast the finite difference schemes in Matrix Form

$$\boxed{\frac{d\vec{u}}{dt} = A\vec{u} - \vec{f}(t)} \quad (1)$$

2. For Example

(a) PDE: Diffusion Eq. ODE:  $\frac{d\vec{u}}{dt} = \frac{\nu}{\Delta x^2} B(1, -2, 1)\vec{u} + (\vec{bc})$

(b) PDE: Convection Eq. ODE:  $\frac{d\vec{u}}{dt} = -\frac{a}{2\Delta x} B_p(-1, 0, 1)\vec{u}$

## General Discussion

1. Note that the elements in the matrix  $A$  depend upon both the PDE and the type of differencing scheme chosen.
2. Vector  $\vec{f}(t)$  determined by the BC and possibly source terms.
3. In general, even the Euler and Navier-Stokes equations can be expressed in the form of Eq. 1.
  - (a) In such cases the equations are nonlinear, that is, the elements of  $A$  depend on the solution  $\vec{u}$  and are usually derived by finding the Jacobian of a flux vector.
  - (b) Although the equations are nonlinear, the linear analysis leads to diagnostics that are surprisingly accurate when used to evaluate many aspects of numerical methods as they apply to the Euler and Navier-Stokes equations.

## General Solution of Coupled ODE's

$$\frac{d\vec{u}}{dt} = A\vec{u} - \vec{f}(t) \quad (2)$$

1. Assume that the  $M \times M$  matrix  $A$  has a complete eigensystem

$$X^{-1}AX = \Lambda$$

2. Diagonalizing Eq. 2

$$X^{-1} \frac{d\vec{u}}{dt} = X^{-1}AX \cdot X^{-1}\vec{u} - X^{-1}\vec{f}(t) \quad (3)$$

3. Eq. 3 can be modified to

$$\frac{d}{dt}X^{-1}\vec{u} = \Lambda X^{-1}\vec{u} - X^{-1}\vec{f}(t) \quad (4)$$

## General Solution of Coupled ODE's

1. Introduce new variables  $\vec{w}$  and  $\vec{g}$  such that

$$\vec{w} = X^{-1}\vec{u} \quad , \quad \vec{g}(t) = X^{-1}\vec{f}(t) \quad (5)$$

2. Reducing Eq. 2 to a new algebraic form

$$\frac{d\vec{w}}{dt} = \Lambda\vec{w} - \vec{g}(t) \quad (6)$$

3. Written line by line: set of independent, single, first-order eqs

$$\begin{aligned} w_1' &= \lambda_1 w_1 - g_1(t) \\ &\vdots \\ w_m' &= \lambda_m w_m - g_m(t) \\ &\vdots \\ w_M' &= \lambda_M w_M - g_M(t) \end{aligned} \quad (7)$$

## General Solution of Coupled ODE's: Continued

1. Equations can be solved separately and then re-coupled, using the inverse of the relations given in Eqs. 5:

$$\vec{u}(t) = X\vec{w}(t) = \sum_{m=1}^M w_m(t)\vec{x}_m$$

where  $\vec{x}_m$  is the  $m^{\text{th}}$  column of  $X$ , corresponding to  $\lambda_m$ .

2. the solution to any line in Eq. 7 is

$$w_m(t) = c_m e^{\lambda_m t} + \frac{1}{\lambda_m} g_m$$

where the  $c_m$  are constants that depend on the initial conditions.

## General Solution of Coupled ODE's: Continued

1. Transforming back to the  $u$ -system gives

$$\begin{aligned}\vec{u}(t) &= X\vec{w}(t) \\ &= \sum_{m=1}^M w_m(t)\vec{x}_m = \sum_{m=1}^M c_m e^{\lambda_m t} \vec{x}_m + \sum_{m=1}^M \frac{1}{\lambda_m} g_m \vec{x}_m \\ &= \sum_{m=1}^M c_m e^{\lambda_m t} \vec{x}_m + X\Lambda^{-1}X^{-1}\vec{f} \\ &= \underbrace{\sum_{m=1}^M c_m e^{\lambda_m t} \vec{x}_m}_{\text{Transient}} + \underbrace{A^{-1}\vec{f}}_{\text{Steady-state}}\end{aligned}\tag{8}$$

## General Solution of Coupled ODE's: Continued

1. Note that the steady-state solution is  $A^{-1}\vec{f}$ , as might be expected.
2. The first group of terms on the right side of this equation is referred to classically as the *complementary solution* or the solution of the homogeneous equations.
3. The second group is referred to classically as the *particular solution* or the particular integral.
4. In our application to fluid dynamics, it is more instructive to refer to these groups as the *transient* and *steady-state* solutions, respectively.

## Eigenspectrum for Model ODE's: Diffusion

1. PDE: Diffusion Eq. ODE:  $\frac{d\vec{u}}{dt} = \frac{\nu}{\Delta x^2} B(1, -2, 1)\vec{u} + (\vec{bc})$

2. From Appendix B in Text: eigenvalues of

$\frac{\nu}{\Delta x^2} B(1, -2, 1), m = 1, 2, \dots, M :$

$$\lambda_m = \frac{\nu}{\Delta x^2} \left[ -2 + 2 \cos \left( \frac{m\pi}{M+1} \right) \right] = \frac{-4\nu}{\Delta x^2} \sin^2 \left( \frac{m\pi}{2(M+1)} \right) \quad (9)$$

3. The Eigenvectors, for  $M = 4$

$$\begin{bmatrix} \sin \left( \frac{\pi}{M+1} \right) & \sin \left( 2 \cdot \frac{\pi}{M+1} \right) & \sin \left( 3 \cdot \frac{\pi}{M+1} \right) & \sin \left( 4 \cdot \frac{\pi}{M+1} \right) \\ \sin \left( \frac{2\pi}{M+1} \right) & \sin \left( 2 \cdot \frac{2\pi}{M+1} \right) & \sin \left( 3 \cdot \frac{2\pi}{M+1} \right) & \sin \left( 4 \cdot \frac{2\pi}{M+1} \right) \\ \sin \left( \frac{3\pi}{M+1} \right) & \sin \left( 2 \cdot \frac{3\pi}{M+1} \right) & \sin \left( 3 \cdot \frac{3\pi}{M+1} \right) & \sin \left( 4 \cdot \frac{3\pi}{M+1} \right) \\ \sin \left( \frac{4\pi}{M+1} \right) & \sin \left( 2 \cdot \frac{4\pi}{M+1} \right) & \sin \left( 3 \cdot \frac{4\pi}{M+1} \right) & \sin \left( 4 \cdot \frac{4\pi}{M+1} \right) \end{bmatrix}$$

## Eigensolution for Diffusion Equation

1. For the diffusion equation, Eq. 8 becomes

$$u_j(t) = \sum_{m=1}^M c_m e^{-\nu \kappa_m^{*2} t} \sin \kappa_m x_j + (A^{-1} f)_j \quad (10)$$

2. With the modified wavenumber for diffusion defined as

$$-\kappa_m^{*2} = \frac{-4\nu}{\Delta x^2} \sin^2 \left( \frac{m\pi}{2(M+1)} \right)$$

3. Comparing with the exact solution to the PDE evaluated at the nodes of the grid, ( $h(x_j)$  particular solution satisfying the BC):

$$u_j(t) = \sum_{m=1}^M c_m e^{-\nu \kappa_m^{*2} t} \sin \kappa_m x_j + h(x_j), \quad (11)$$

4. The solutions are identical except for the steady solution and the modified wavenumber in the transient term.
5. The modified wavenumber is an approximation to the actual wavenumber.
6. The difference causes the various modes (or eigenvector components) to decay at rates which differ from the exact solution.
7. With conventional differencing schemes, low wavenumber modes are accurately represented, while high wavenumber modes (if they have significant amplitudes) can have large errors.

## Eigenspectrum Model ODE's: Periodic Convection

1. PDE: Convection Eq. ODE:

$$\frac{d\vec{u}}{dt} = -\frac{a}{2\Delta x} B_p(-1, 0, 1)\vec{u}; m = 0, 1, 2, \dots, M - 1$$

2. From Appendix B in Text: eigenvalues of  $-\frac{a}{2\Delta x} B_p(-1, 0, 1)$ :

$$\lambda_m = \frac{-ia}{\Delta x} \sin\left(\frac{2m\pi}{M}\right) = -i\kappa_m^* a \quad (12)$$

3. The right-hand eigenvectors are given by

$$\vec{x}_m = e^{ij(2\pi m/M)}, \quad \begin{array}{l} j = 0, 1, \dots, M - 1 \\ m = 0, 1, \dots, M - 1 \end{array}$$

With  $x_j = j \cdot \Delta x = j \cdot 2\pi/M$

## Eigensolution for Convection Equation

1. For the biconvection equation the ODE solution as

$$u_j(t) = \sum_{m=0}^{M-1} c_m e^{-i\kappa_m^* at} e^{i\kappa_m x_j}, j = 0, 1, \dots, M - 1 \quad (13)$$

2. Comparing the exact solution of the PDE evaluated at the nodes of the grid:

$$u_j(t) = \sum_{m=0}^{M-1} f_m(0) e^{-i\kappa_m at} e^{i\kappa_m x_j}, j = 0, 1, \dots, M - 1 \quad (14)$$

3. This Should look very familiar from the previous lecture on Modified Wave Numbers

4. Once again the difference appears through the modified wavenumber,  $k^*$
5. This leads to an error in the speed with which various modes are convected.
6. In the case of non-centered the modified wavenumber is complex producing nonphysical decay or growth in the numerical solution.

## Representative ODE

1. Our next objective is to find a “typical” single ODE to analyze.
2. We found the uncoupled solution to a set of ODE’s above with a typical member of the family

$$\frac{dw_m}{dt} = \lambda_m w_m - g_m(t) \quad (15)$$

3. The goal in our analysis is to study typical behavior of general situations, not particular problems.
4. The role of  $\lambda_m$  is clear; it stands for some representative eigenvalue in the original  $A$  matrix.

5. What should we use for  $g_m(t)$  when the time dependence cannot be ignored?
6. One can express any one of the forcing terms  $g_m(t)$  as a finite Fourier series.
7. For example  $-g(t) = \sum_k a_k e^{ikt}$
8. Eq. 15 has the exact solution:

$$w(t) = ce^{\lambda t} + \sum_k \frac{a_k e^{ikt}}{ik - \lambda}$$

9. From this we can extract the  $k$ th term and replace  $ik$  with  $\mu$ .

This leads to

The Representative ODE

$$\frac{dw}{dt} = \lambda w + ae^{\mu t}$$

(16)

10. The exact solution of the representative ODE is (for  $\mu \neq \lambda$ ):

$$w(t) = ce^{\lambda t} + \frac{ae^{\mu t}}{\mu - \lambda}$$

(17)