

Chapter 8

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Stiffness Definition for ODE's

- “Stiffness” comes about from the numerical analysis of mathematical models constructed to simulate dynamic phenomena containing widely different time scales.
- Assume that our CFD problem is modeled with sufficient accuracy by a coupled set of ODE's producing an A matrix.
- The difference between the dynamic scales in physical space is represented by the difference in the magnitude of the eigenvalues of A .
- Consider now the form of the exact solution of a system of ODE's with a complete eigensystem, \vec{x}_m and λ_m .

Driving and Parasitic Eigenvalues

- Eigenvalues of A will be complex with negative real parts.
- Ordering the eigenvalues by their magnitudes

$$|\lambda_1| \leq |\lambda_2| \leq \dots \leq |\lambda_M| \quad (1)$$

- Subdivide the transient solution into two parts.

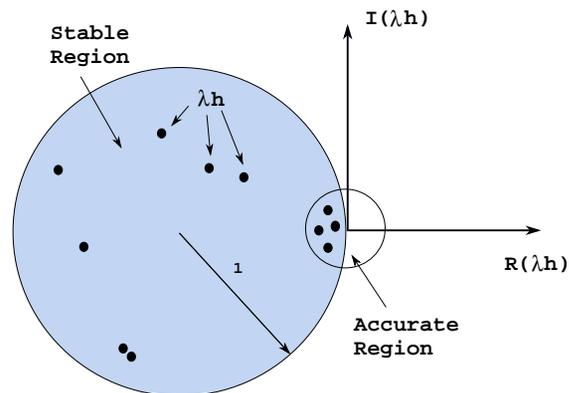
$$\begin{array}{l} \text{Transient} \\ \text{Solution} \end{array} = \underbrace{\sum_{m=1}^p c_m e^{\lambda_m t} \vec{x}_m}_{\text{Driving}} + \underbrace{\sum_{m=p+1}^M c_m e^{\lambda_m t} \vec{x}_m}_{\text{Parasitic}} \quad (2)$$

- Separate our eigenvalue spectrum into two groups
 - $[\lambda_1 \rightarrow \lambda_p]$ called the driving eigenvalues

- Choice of a time-step and marching method must accurately approximate the time variation of the associated eigenvectors
- $[\lambda_{p+1} \rightarrow \lambda_M]$, called the parasitic eigenvalues
- Time accuracy is not required for the eigenvectors associated with these
- Their presence must not contaminate the accuracy.
- Time accuracy dictated by the driving eigenvalues
- Numerical stability requirements by the parasitic ones.

Resolution and Stability

- In many numerical applications, eigenvectors associated
 - with the small $|\lambda_m|$ are well resolved
 - with the large $|\lambda_m|$ are resolved much less accurately.
- In the complex λh plane



- Time step chosen so that time accuracy for the eigenvectors associated with the eigenvalues lying in the small circle

- Stability without time accuracy is associated with the eigenvalues lying outside of the small circle but still inside the large circle.

The whole concept of stiffness in CFD arises from the fact that we often do not need the *time resolution* of eigenvectors associated with the large $|\lambda_m|$ in the transient solution, although these eigenvectors must remain coupled into the system to maintain a high accuracy of the *spatial resolution*.

Stiffness Classifications

- An inherently stable set of ODE's is stiff if

$$|\lambda_p| \ll |\lambda_M|$$

- Define the ratio

$$C_r = |\lambda_M| / |\lambda_p|$$

- For example, categories

Mildly-stiff		C_r	$< 10^2$
Strongly-stiff	$10^3 <$	C_r	$< 10^5$
Extremely-stiff	$10^6 <$	C_r	$< 10^8$
Pathologically-stiff	$10^9 <$	C_r	

- Bounds are arbitrary.

Relation of Stiffness to Space Mesh Size

- Many flow fields are characterized by a few regions having high spatial gradients of the dependent variables and other domains having relatively low gradient phenomena.
- Diffusion equation: eigenvalues are all real negative numbers
- Consider the case when *all* of the eigenvalues are parasitic, i.e. Steady State

- Let M be the number of cells, $\Delta x = \frac{\pi}{M+1}$,

$$\lambda_m = -\frac{4\nu}{\Delta x^2} \sin^2 \left(\frac{m\pi}{2(M+1)} \right)$$

$$\lambda_1 = -\frac{4\nu}{\Delta x^2} \sin^2 \left(\frac{\pi}{2(M+1)} \right) \approx -\left(\frac{4\nu}{\Delta x^2} \right) \left(\frac{\Delta x}{2} \right)^2 = -\nu$$

$$\lambda_M \approx -\frac{4\nu}{\Delta x^2} \sin^2 \left(\frac{\pi}{2} \right) = -\frac{4\nu}{\Delta x^2}$$

- Stiffness ratio, C_r

$$|\lambda_M| / |\lambda_1| \approx \frac{4}{\Delta x^2} = 4 \left(\frac{M+1}{\pi} \right)^2$$

- Note C_r increases as $\frac{1}{(\Delta x)^2}$

Stiffness For Convection

- Convection equation: periodic central differences
- Let M be the number of cells, $\Delta x = \frac{\pi}{M}$,

$$\lambda_m = -\frac{a}{\Delta x} i \sin m\Delta x$$

$$\lambda_1 = -\frac{a}{\Delta x} i \sin \Delta x \approx -i \frac{a}{\Delta x} (\Delta x) = -ia$$

$$\lambda_M = -i \frac{a}{\Delta x} \sin M\Delta x \approx -iaM$$

- Stiffness ratio, C_r

$$|\lambda_M| / |\lambda_1| \approx M = \frac{\pi}{\Delta x} \approx \frac{1}{\Delta x}$$

- Note C_r increases as $\frac{1}{\Delta x}$

Coping With Stiffness

- Mildly-stiff system composed of a coupled two-equation set having the two eigenvalues $\lambda_1 = -100$ and $\lambda_2 = -1$.
- For the ODE solution

$$u_1(t) = c_1 e^{-100t} x_{11} + c_2 e^{-t} x_{12} + (PS)_1$$

$$u_2(t) = c_1 e^{-100t} x_{21} + c_2 e^{-t} x_{22} + (PS)_2$$

- The first components decay rapidly (as e^{-100t}) relative to the second components, (e^{-t})

Example using Euler Explicit

- Explicit Euler method applied to the representative equation.
 - The transient solution is $u_n = (1 + \lambda h)^n$
 - h is chosen so that integration will be numerically stable.

$$u_1(n) = c_1(1 - 100h)^n x_{11} + c_2(1 - h)^n x_{12} + (PS)_1$$

$$u_2(n) = c_1(1 - 100h)^n x_{21} + c_2(1 - h)^n x_{22} + (PS)_2$$

- Assume that our accuracy requirements are such that sufficient accuracy is obtained as long as $|\lambda h| \leq 0.1$.
- Defines a time step limit based on *accuracy* considerations of $h = 0.001$ for λ_1 and $h = 0.1$ for λ_2 .
- The time step limit is $h = 0.02$ is based on *stability* for λ_1 .

- Let $c_1 = c_2 = 1$ and assume that an amplitude less than 0.001 is *negligible*.
- 66 time steps with $h = 0.001$ to resolve the λ_1 term.
- With this time step the λ_2 term is resolved exceedingly well.
- After 66 steps
 - Amplitude: λ_1 term (i.e., $(1 - 100h)^n$) is less than 0.001
 - Amplitude: λ_2 term (i.e., $(1 - h)^n$) is 0.9361.
- Hence the λ_1 term can now be considered negligible.
- To drive the $(1 - h)^n$ term to zero (i.e., below 0.001), we would like to change the step size to $h = 0.1$ and continue.

- This is not possible because of the coupled presence of $(1 - 100h)^n$, which in just 10 steps at $h = 0.1$ amplifies those terms by $\approx 10^9$
- In fact, with $h = 0.02$, the maximum step size that can be taken in order to maintain stability, about 339 time steps have to be computed in order to drive e^{-t} to below 0.001.
- Thus the total simulation requires 405 time steps.

Implicit Methods

- Unconditionally stable implicit trapezoidal method.
- In this case $\sigma = \frac{1+\lambda h}{1-\lambda h}$
- For the previous example: $\lambda_1 = -100, \lambda_2 = -1$.

$$u_1(n) = c_1 \left(\frac{1 - 50h}{1 + 50h} \right)^n x_{11} + c_2 \left(\frac{1 - 0.5h}{1 + 0.5h} \right)^n x_{12} + (PS)_1$$

$$u_2(n) = c_1 \left(\frac{1 - 50h}{1 + 50h} \right)^n x_{21} + c_2 \left(\frac{1 - 0.5h}{1 + 0.5h} \right)^n x_{22} + (PS)_2$$

- Using $h = 0.001$ will resolve the initial transient of the term e^{-100t}

- After 70 time steps
 - The λ_1 term $\left(\frac{1-50h}{1+50h}\right)^{70} = 0.0009 < 0.001$: negligible.
 - The λ_2 term $\left(\frac{1-0.5h}{1+0.5h}\right)^{70} = 0.9324$, not negligible.
- Proceed to calculate the remaining part of the event using our desired step size $h = 0.1$ without any problem of instability on either term.
- After 69 more steps the amplitude of the second term ≈ 0.001 .
- In both intervals the desired solution is second-order accurate and well resolved.
- The total simulation requires 139 time steps.