Lecture 7

Numerical Dissipation

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Quick Review

Quick Review

Semi-discrete Approach

- PDE converted to system of ODEs by spatial discretization
- ODEs converted to OΔEs by time-marching method

Spatial Operator Matrices

- periodic linear convection equation leads to $aB_{\mathrm{p}}(-1, 0, 1)/2\Delta x$
  - pure imaginary eigenvalues
- diffusion equation leads to $\nu B(1, -2, 1)/\Delta x^2$
  - negative real eigenvalues

Introduction

Introduction

Numerical Dissipation

Needed for

- stability and convergence
- damping of under-resolved spurious high-frequency modes
- avoiding oscillations at discontinuities such as shocks
- preservation of positivity of quantities that must be positive

Not needed if scales relevant to physical dissipative processes are sufficiently well resolved, for example direct numerical simulations of turbulent flows

Introduction

Numerical Dissipation

Introduced through

- upwinding
- artificial dissipation

One-sided First-Derivative Space Differencing

One-sided First-Derivative Space Differencing

Linear Convection Equation

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x}$$

Point Operator

$$\begin{aligned} -a(\delta_x u)_j &= \frac{-a}{2\Delta x}[-(1+\beta)u_{j-1} + 2\beta u_j + (1-\beta)u_{j+1}] \\ &= \frac{-a}{2\Delta x}[(-u_{j-1} + u_{j+1}) + \beta(-u_{j-1} + 2u_j - u_{j+1})] \end{aligned}$$

A first-order *backward* difference operator is given by $\beta = 1$, and a first-order *forward* difference operator is given by $\beta = -1$.

One-sided First-Derivative Space Differencing

Matrix Operator

Matrix Operator for Periodic Boundary Conditions

$$-a\delta_x = \frac{-a}{2\Delta x} B_{\mathrm{p}}(-1-\beta, 2\beta, 1-\beta)$$

Eigenvalues of the matrix

$$\lambda_m = \frac{-a}{\Delta x}\left\{\beta\left[1-\cos\left(\frac{2\pi m}{M}\right)\right] + \mathrm{i}\sin\left(\frac{2\pi m}{M}\right)\right\}, \quad m = 0, 1, \ldots, M-1$$

If a is positive, the forward difference operator ($\beta = -1$) produces $\Re(\lambda_m) > 0$, the centered difference operator ($\beta = 0$) produces $\Re(\lambda_m) = 0$, and the backward difference operator produces $\Re(\lambda_m) < 0$. Hence the forward difference operator is inherently unstable, while the centered and backward operators are inherently stable. If a is negative, the roles are reversed. But the eigenvalues of the 1D Euler equations $(u, u+a, u-a)$ are of mixed sign in subsonic flow.

Upwind Schemes

Upwind Schemes

Linear Convection Equation (scalar)

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + (a^+ + a^-)\frac{\partial u}{\partial x} = 0 \quad , \quad a^{\pm} = \frac{a \pm |a|}{2}$$

If $a \geq 0$, then $a^+ = a \geq 0$ and $a^- = 0$. Alternatively, if $a \leq 0$, then $a^+ = 0$ and $a^- = a \leq 0$. Now for the a^+ (≥ 0) term we can safely backward difference and for the a^- (≤ 0) term forward difference.

Equivalent form:

Point Operator

$$-a(\delta_x u)_j = \frac{-1}{2\Delta x}[a(-u_{j-1} + u_{j+1}) + |a|(-u_{j-1} + 2u_j - u_{j+1})]$$

Flux-Vector Splitting

Linear, Constant-Coefficient Hyperbolic System

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = \frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = 0$$

For a hyperbolic system, A is diagonalizable:

$$\Lambda = X^{-1}AX$$

where Λ is a diagonal matrix containing the eigenvalues of A, and X is the matrix of right eigenvectors, and the eigenvalues are all real.

Flux-Vector Splitting

Premultiplying by X^{-1}, postmultiplying A by the product XX^{-1}, and noting that X and X^{-1} are constants, we obtain

$$\frac{\partial X^{-1}u}{\partial t} + \frac{\partial \overbrace{X^{-1}AX}^{\Lambda} X^{-1}u}{\partial x} = 0 \ .$$

With $w = X^{-1}u$, this can be rewritten as

$$\frac{\partial w}{\partial t} + \Lambda \frac{\partial w}{\partial x} = 0 \ .$$

When written in this manner, the equations have been decoupled into scalar equations of the form

$$\frac{\partial w_i}{\partial t} + \lambda_i \frac{\partial w_i}{\partial x} = 0 \ .$$

Flux-Vector Splitting

Start with the decoupled equations and use the same approach we used for the linear convection equation:

$$\frac{\partial w_i}{\partial t} + \lambda_i \frac{\partial w_i}{\partial x} = 0$$

$$\Lambda = \Lambda^+ + \Lambda^-$$

$$\Lambda^+ = \frac{\Lambda + |\Lambda|}{2}, \qquad \Lambda^- = \frac{\Lambda - |\Lambda|}{2}$$

$$\frac{\partial w}{\partial t} + \Lambda \frac{\partial w}{\partial x} = \frac{\partial w}{\partial t} + \Lambda^+ \frac{\partial w}{\partial x} + \Lambda^- \frac{\partial w}{\partial x} = 0$$

Flux-Vector Splitting

Now let's recouple the equations:

$$\frac{\partial Xw}{\partial t} + \frac{\partial X\Lambda^+X^{-1}Xw}{\partial x} + \frac{\partial X\Lambda^-X^{-1}Xw}{\partial x} = 0$$

$$A^+ = X\Lambda^+X^{-1}, \qquad A^- = X\Lambda^-X^{-1}$$

$$\frac{\partial u}{\partial t} + \frac{\partial A^+u}{\partial x} + \frac{\partial A^-u}{\partial x} = 0$$

Define

$$f^+ = A^+u, \qquad f^- = A^-u$$

Flux-Vector Splitting

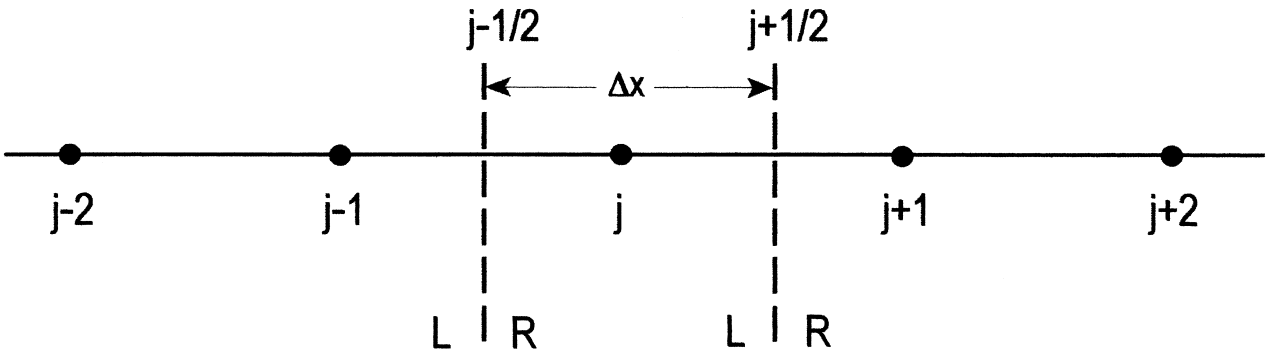
$$\frac{\partial u}{\partial t} + \frac{\partial f^+}{\partial x} + \frac{\partial f^-}{\partial x} = 0$$

$$f = f^+ + f^-$$

Use a backward difference for the f^+ term and a forward difference for the f^- term

Flux-Difference Splitting

Finite Volume Method



Control volume in one dimension

Apply a reconstruction in each cell to get left and right states at each interface

Find a single numerical flux at the interface based on these states

Flux-Difference Splitting

Scalar Problem (decoupled)

$$\frac{\partial w}{\partial t} + \Lambda \frac{\partial w}{\partial x} = 0$$

Centered flux function (nondissipative):

$$\hat{g}_{j+1/2} = \frac{1}{2}[g(w^{\mathrm{L}}) + g(w^{\mathrm{R}})]$$

Upwind flux function (dissipative):

$$(\hat{g}_i)_{j+1/2} = \begin{cases} \lambda_i (w_i)^{\mathrm{L}} \text{ if } \lambda_i > 0 \\ \lambda_i (w_i)^{\mathrm{R}} \text{ if } \lambda_i < 0 \end{cases}$$

$$(\hat{g}_i)_{j+1/2} = \frac{1}{2}\lambda_i \left[(w_i)^{\mathrm{L}} + (w_i)^{\mathrm{R}}\right] + \frac{1}{2}|\lambda_i| \left[(w_i)^{\mathrm{L}} - (w_i)^{\mathrm{R}}\right]$$

Flux-Difference Splitting

Now let's recouple

$$\hat{g}_{j+1/2} = \frac{1}{2}\Lambda\left(w^{\mathrm{L}} + w^{\mathrm{R}}\right) + \frac{1}{2}|\Lambda|\left(w^{\mathrm{L}} - w^{\mathrm{R}}\right)$$

$$X\hat{g}_{j+1/2} = \frac{1}{2}X\Lambda X^{-1}X\left(w^{\mathrm{L}} + w^{\mathrm{R}}\right) + \frac{1}{2}X|\Lambda|X^{-1}X\left(w^{\mathrm{L}} - w^{\mathrm{R}}\right)$$

$$\hat{f}_{j+1/2} = \frac{1}{2}\left(f^{\mathrm{L}} + f^{\mathrm{R}}\right) + \frac{1}{2}|A|\left(u^{\mathrm{L}} - u^{\mathrm{R}}\right)$$

$$|A| = X|\Lambda|X^{-1}$$

Flux-Difference Splitting

For full upwinding, for example in supersonic flow where all eigenvalues have the same sign

$$\hat{f}_{j+1/2} = \begin{cases} f^{\mathrm{L}} \text{ if all } \lambda_i\text{'s} > 0 \\ f^{\mathrm{R}} \text{ if all } \lambda_i\text{'s} < 0 \end{cases}$$

Achieved with $A_{j+1/2} = A(u_{j+1/2})$

$$\hat{f}_{j+1/2} = \frac{1}{2}\left(f^{\mathrm{L}} + f^{\mathrm{R}}\right) + \frac{1}{2}|A_{j+1/2}|\left(u^{\mathrm{L}} - u^{\mathrm{R}}\right)$$

where $A_{j+1/2}$ satisfies (Rankine-Hugoniot jump condition across shocks)

$$f^{\mathrm{L}} - f^{\mathrm{R}} = A_{j+1/2}\left(u^{\mathrm{L}} - u^{\mathrm{R}}\right)$$

Flux-Difference Splitting

This property of $A_{j+1/2}$ is obtained from the following Roe-average state:

$$
\begin{aligned}
u_{j+1/2} &= \frac{\sqrt{\rho^{\mathrm{L}}}u^{\mathrm{L}} + \sqrt{\rho^{\mathrm{R}}}u^{\mathrm{R}}}{\sqrt{\rho^{\mathrm{L}}} + \sqrt{\rho^{\mathrm{R}}}} \\
H_{j+1/2} &= \frac{\sqrt{\rho^{\mathrm{L}}}H^{\mathrm{L}} + \sqrt{\rho^{\mathrm{R}}}H^{\mathrm{R}}}{\sqrt{\rho^{\mathrm{L}}} + \sqrt{\rho^{\mathrm{R}}}}
\end{aligned}
$$

Artificial Dissipation

Artificial Dissipation

Any operator can be split into its antisymmetric part and its symmetric part, for example for first-order backward differencing:

$$\left(\delta_x^{\mathrm{a}} u\right)_j = \frac{u_{j+1} - u_{j-1}}{2\Delta x}, \qquad \left(\delta_x^{\mathrm{s}} u\right)_j = \frac{-u_{j+1} + 2u_j - u_{j-1}}{2\Delta x}$$

Phase error comes from the antisymmetric part, dissipation from the symmetric part

Consistent with what we saw before, this can be applied to the scalar linear convection equation with wave speed λ_i of arbitrary sign as follows:

$$\lambda_i \delta_x = \lambda_i \delta_x^{\mathrm{a}} + |\lambda_i| \delta_x^{\mathrm{s}}$$

Artificial Dissipation

Extension to hyperbolic systems follows the same recoupling approach we used for upwinding:

$$\delta_x(Au) = \delta_x^{\mathrm{a}}(Au) + \delta_x^{\mathrm{s}}(|A|u)$$

leading to

$$\delta_x f = \delta_x^{\mathrm{a}} f + \delta_x^{\mathrm{s}}(|A|u)$$

A common choice for the symmetric operator is

$$(\delta_x^{\mathrm{s}} u)_j = \frac{\epsilon}{\Delta x}(u_{j-2} - 4u_{j-1} + 6u_j - 4u_{j+1} + u_{j+2})$$

which is proportional to $\Delta x^3 u_{xxxx}$, so it provides a dissipative term that is third order, which can be combined with second-order centered differencing

Upwind-Biased Schemes

Example: Third-order upwind-biased operator split into antisymmetric and symmetric parts:

$$
\begin{aligned}
(\delta_x u)_j &= \frac{1}{6\Delta x}(u_{j-2} - 6u_{j-1} + 3u_j + 2u_{j+1}) \\
&= \frac{1}{12\Delta x}[(u_{j-2} - 8u_{j-1} + 8u_{j+1} - u_{j+2}) \\
&\quad +(u_{j-2} - 4u_{j-1} + 6u_j - 4u_{j+1} + u_{j+2})] \ .
\end{aligned}
$$

The antisymmetric component of this operator is the fourth-order centered difference operator. The symmetric component approximates $\Delta x^3 u_{xxxx}/12$. Therefore, this operator produces fourth-order accuracy in phase with a third-order dissipative term.