Lectures 9-11

An Implicit Finite-Difference Algorithm for the Euler and Navier-Stokes Equations

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The Euler and Navier-Stokes Equations

Partial Differential Equation Form

$$\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} = \frac{\partial E_{\mathrm{v}}}{\partial x} + \frac{\partial F_{\mathrm{v}}}{\partial y}$$

$$Q = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ e \end{bmatrix}, \quad E = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ u(e+p) \end{bmatrix}, \quad F = \begin{bmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ v(e+p) \end{bmatrix}$$

$$E_{\mathrm{v}} = \begin{bmatrix} 0 \\ \tau_{xx} \\ \tau_{xy} \\ f_4 \end{bmatrix}, \quad F_{\mathrm{v}} = \begin{bmatrix} 0 \\ \tau_{xy} \\ \tau_{yy} \\ g_4 \end{bmatrix}$$

Partial Differential Equation Form

Total energy per unit volume

$$e = \rho \left(\epsilon + \frac{u^2 + v^2}{2} \right)$$

Equation of state (thermally and calorically perfect gas)

$$p = \rho R T$$

Internal energy per unit mass

$$\epsilon = c_{\mathrm{v}} T$$

Partial Differential Equation Form

Pressure in terms of conservative variables

$$
\begin{aligned}
p &= \rho R T \\
&= \rho R \left(\frac{\epsilon}{c_{\mathrm{v}}} \right) \\
&= (\gamma - 1)\rho\epsilon \\
&= (\gamma - 1) \left(e - \frac{\rho}{2}(u^2 + v^2) \right) \\
&= (\gamma - 1) \left[e - \frac{1}{2\rho} \left((\rho u)^2 + (\rho v)^2 \right) \right]
\end{aligned}
$$

where we've used the relation

$$
c_{\mathrm{v}} = \frac{R}{\gamma - 1}
$$

Partial Differential Equation Form

Speed of sound (perfect gas)

$$a^2 = \frac{\gamma p}{\rho} = \gamma R T$$

Viscous tress tensor for a Newtonian fluid

$$\begin{aligned}
\tau_{xx} &= \mu \left(\frac{4}{3} \frac{\partial u}{\partial x} - \frac{2}{3} \frac{\partial v}{\partial y} \right) \\
\tau_{xy} &= \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\
\tau_{yy} &= \mu \left(-\frac{2}{3} \frac{\partial u}{\partial x} + \frac{4}{3} \frac{\partial v}{\partial y} \right)
\end{aligned}$$

$\mu(T)$ from Sutherland's law

Partial Differential Equation Form

Fourier's law of heat conduction

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)$$

$$Pr=\frac{\mu c_{\mathrm{p}}}{k}=0.71 \quad (0.9 \text{ for turbulent flow})$$

$$\frac{\partial}{\partial x}\left(\frac{\mu}{Pr(\gamma-1)}\frac{\partial a^2}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\mu}{Pr(\gamma-1)}\frac{\partial a^2}{\partial y}\right)$$

with

$$c_{\mathrm{p}}=\frac{\gamma R}{\gamma-1}$$

Viscous and heat conduction terms in energy equation

$$
\begin{aligned}
f_4 &= u\tau_{xx} + v\tau_{xy} + \frac{\mu}{Pr(\gamma-1)}\frac{\partial a^2}{\partial x} \\
g_4 &= u\tau_{xy} + v\tau_{yy} + \frac{\mu}{Pr(\gamma-1)}\frac{\partial a^2}{\partial y}
\end{aligned}
$$

Nondimensionalization

$$
\begin{aligned}
\widetilde{x} &= \frac{x}{l}, \qquad \widetilde{y} = \frac{y}{l}, \qquad \widetilde{t} = \frac{t a_\infty}{l} \\
\widetilde{\rho} &= \frac{\rho}{\rho_\infty}, \qquad \widetilde{u} = \frac{u}{a_\infty}, \qquad \widetilde{v} = \frac{v}{a_\infty} \\
\widetilde{e} &= \frac{e}{\rho_\infty a_\infty^2}, \qquad \widetilde{\mu} = \frac{\mu}{\mu_\infty}
\end{aligned}
$$

$$
Re = \frac{\rho_\infty l a_\infty}{\mu_\infty}
$$

Dropping the tildes:

$$
\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} = Re^{-1} \left(\frac{\partial E_{\mathrm{v}}}{\partial x} + \frac{\partial F_{\mathrm{v}}}{\partial y} \right)
$$

Nonconservative form (primitive variables)

$$\frac{\partial R}{\partial t} + \tilde{A}\frac{\partial R}{\partial x} = 0, \qquad R = [\rho, u, p]^T$$

$$\tilde{A} = \begin{bmatrix} u & \rho & 0 \\ 0 & u & \rho^{-1} \\ 0 & \gamma p & u \end{bmatrix}$$

$$\text{conservative form :} \qquad \frac{\partial}{\partial x}(\rho u)$$

$$\text{nonconservative form :} \qquad \rho\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x}$$

Flux Jacobian

$$\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} = 0$$

$$Q = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} = \begin{bmatrix} \rho \\ \rho u \\ e \end{bmatrix}, \quad E = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ u(e+p) \end{bmatrix}$$

$$\frac{\partial Q}{\partial t} + A\frac{\partial Q}{\partial x} = 0$$

$$A = \frac{\partial E}{\partial Q}$$

Flux Jacobian

$$E = \begin{bmatrix} Q_2 \\ (\gamma - 1)Q_3 + \frac{3-\gamma}{2}\frac{Q_2^2}{Q_1} \\ \gamma\frac{Q_3 Q_2}{Q_1} - \frac{\gamma-1}{2}\frac{Q_2^3}{Q_1^2} \end{bmatrix}$$

$$A = \frac{\partial E_i}{\partial Q_j} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{\gamma-3}{2}\left(\frac{Q_2}{Q1}\right)^2 & (3-\gamma)\frac{Q_2}{Q_1} & \gamma - 1 \\ A_{31} & A_{32} & \gamma\left(\frac{Q_2}{Q_1}\right) \end{bmatrix}$$

Flux Jacobian

$$
\begin{aligned}
A_{31} &= (\gamma - 1)\left(\frac{Q_2}{Q_1}\right)^3 - \gamma\left(\frac{Q_3}{Q_1}\right)\left(\frac{Q_2}{Q_1}\right) \\
A_{32} &= \gamma\left(\frac{Q_3}{Q_1}\right) - \frac{3(\gamma-1)}{2}\left(\frac{Q_2}{Q_1}\right)^2
\end{aligned}
$$

$$
A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{\gamma-3}{2}u^2 & (3-\gamma)u & \gamma - 1 \\ A_{31} & A_{32} & \gamma u \end{bmatrix}
$$

Flux Jacobian

$$
\begin{aligned}
A_{31} &= (\gamma - 1)u^3 - \gamma \frac{ue}{\rho} \\
A_{32} &= \gamma \frac{e}{\rho} - \frac{3(\gamma - 1)}{2} u^2
\end{aligned}
$$

Integral Form

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\iint_{V(t)} Q\mathrm{d}x\mathrm{d}y \quad &+ \quad \oint_{S(t)} (E\mathrm{d}y - F\mathrm{d}x) \\ &= \quad Re^{-1}\oint_{S(t)} (E_\mathrm{v}\mathrm{d}y - F_\mathrm{v}\mathrm{d}x)\end{aligned}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} Q\mathrm{d}V + \oint_{S(t)} \hat{n}\cdot\mathcal{F}\mathrm{d}S = 0$$

$$\mathcal{F} = (E - Re^{-1}E_\mathrm{v})\hat{i} + (F - Re^{-1}F_\mathrm{v})\hat{j}$$

$$\hat{n}\mathrm{d}S = \hat{i}\mathrm{d}y - \hat{j}\mathrm{d}x$$

Physical Boundary Conditions

Inviscid flow (Euler equations)

At a solid wall

$$(u\hat{i} + v\hat{j}) \cdot \hat{n} = 0$$

Viscous flow (Navier-Stokes equations)

At a solid wall

$$u = v = 0$$

$$\nabla T \cdot \hat{n} = 0 \quad \text{(adiabatic)}$$

or isothermal (T specified)

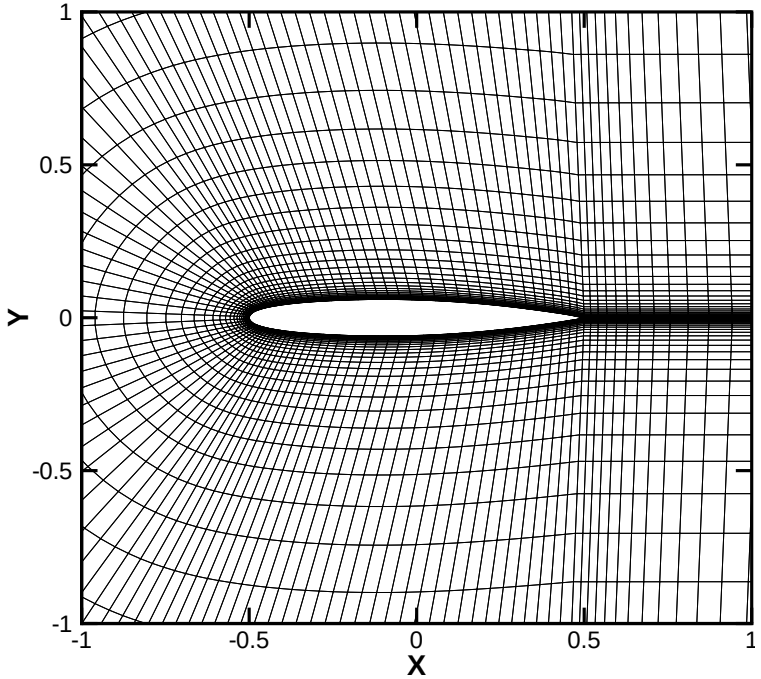
An Implicit Finite-Difference Algorithm for the Euler and Navier-Stokes Equations

Key Characteristics

- node-based data storage; the numerical solution for the state variables is associated with the nodes of the grid
- second-order finite-difference spatial discretization; centered with added numerical dissipation; a simple shock-capturing device
- transformation to generalized curvilinear coordinates; applicable to structured grids
- implicit time marching based on approximate factorization of the resulting matrix operator

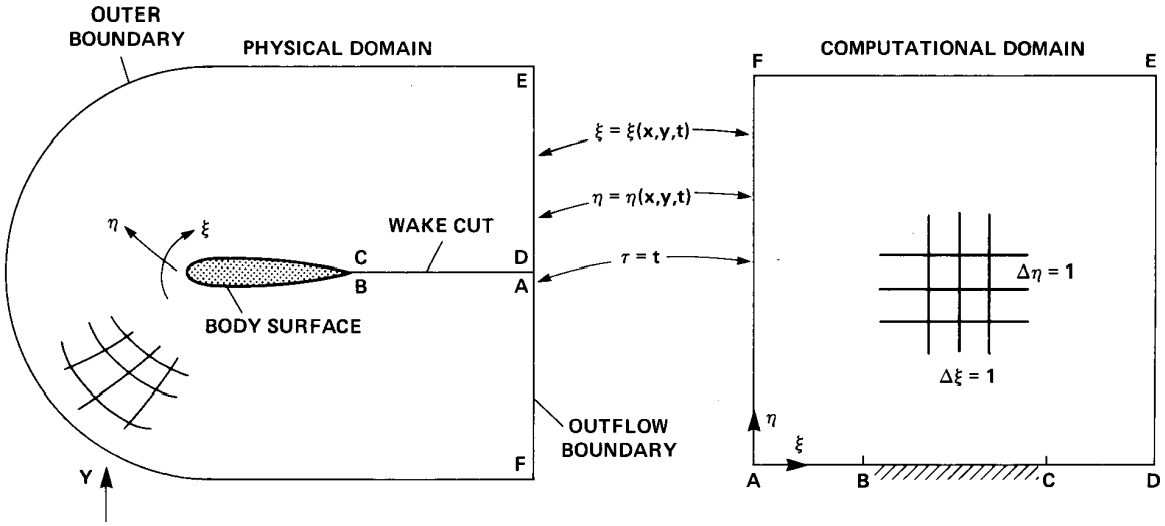
Generalized Curvilinear Coordinate Transformation

Generalized Curvilinear Coordinate Transformation



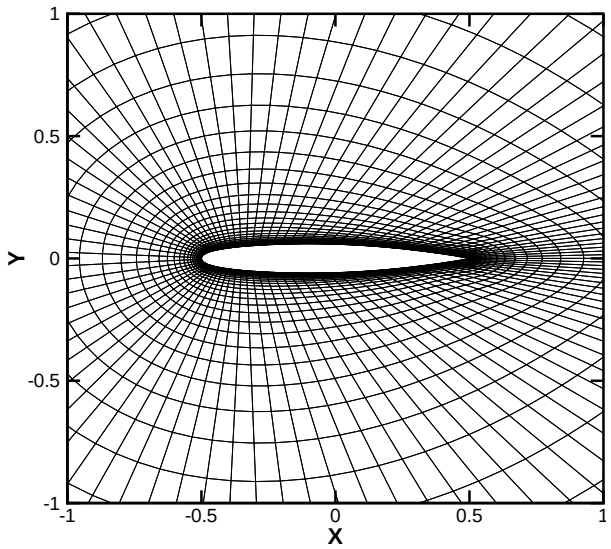
A sample airfoil grid with a "C" topology showing only the region near the airfoil

Generalized Curvilinear Coordinate Transformation



An example of a generalized curvilinear coordinate transformation for a C-mesh

Generalized Curvilinear Coordinate Transformation



A sample airfoil grid with an "O" topology showing only the region near the airfoil

Generalized Curvilinear Coordinate Transformation

$$
\begin{aligned}
\tau &= t \\
\xi &= \xi(x, y, t) \\
\eta &= \eta(x, y, t)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} \\
\frac{\partial}{\partial y} &= \frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta} \\
\frac{\partial}{\partial t} &= \frac{\partial}{\partial \tau} + \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta}
\end{aligned}
$$

Generalized Curvilinear Coordinate Transformation

$$\partial_x \equiv \frac{\partial}{\partial x} \quad \text{and} \quad \xi_x \equiv \frac{\partial \xi}{\partial x}$$

$$\begin{bmatrix} \partial_t \\ \partial_x \\ \partial_y \end{bmatrix} = \begin{bmatrix} 1 & \xi_t & \eta_t \\ 0 & \xi_x & \eta_x \\ 0 & \xi_y & \eta_y \end{bmatrix} \begin{bmatrix} \partial_\tau \\ \partial_\xi \\ \partial_\eta \end{bmatrix}$$

$$\begin{aligned} \partial_\tau Q \quad + \quad & \xi_t \partial_\xi Q + \eta_t \partial_\eta Q + \xi_x \partial_\xi E + \eta_x \partial_\eta E + \xi_y \partial_\xi F + \eta_y \partial_\eta F = \\ & Re^{-1} \left(\xi_x \partial_\xi E_\mathrm{v} + \eta_x \partial_\eta E_\mathrm{v} + \xi_y \partial_\xi F_\mathrm{v} + \eta_y \partial_\eta F_\mathrm{v} \right) \end{aligned}$$

Metric Relations

$$
\begin{aligned}
t &= \tau \\
x &= x(\xi, \eta, \tau) \\
y &= y(\xi, \eta, \tau)
\end{aligned}
$$

$$
\partial_\tau = \partial_t + x_\tau \partial_x + y_\tau \partial_y, \quad \partial_\xi = x_\xi \partial_x + y_\xi \partial_y, \quad \partial_\eta = x_\eta \partial_x + y_\eta \partial_y
$$

$$
\begin{bmatrix} \partial_\tau \\ \partial_\xi \\ \partial_\eta \end{bmatrix} = \begin{bmatrix} 1 & x_\tau & y_\tau \\ 0 & x_\xi & y_\xi \\ 0 & x_\eta & y_\eta \end{bmatrix} \begin{bmatrix} \partial_t \\ \partial_x \\ \partial_y \end{bmatrix}
$$

Metric Relations

$$
\begin{bmatrix} 1 & \xi_t & \eta_t \\ 0 & \xi_x & \eta_x \\ 0 & \xi_y & \eta_y \end{bmatrix} = \begin{bmatrix} 1 & x_\tau & y_\tau \\ 0 & x_\xi & y_\xi \\ 0 & x_\eta & y_\eta \end{bmatrix}^{-1}
$$

$$
= J \begin{bmatrix} (x_\xi y_\eta - y_\xi x_\eta) & (-x_\tau y_\eta + y_\tau x_\eta) & (x_\tau y_\xi - y_\tau x_\xi) \\ 0 & y_\eta & -y_\xi \\ 0 & -x_\eta & x_\xi \end{bmatrix}
$$

$$
\begin{aligned}
\xi_t = J(-x_\tau y_\eta + y_\tau x_\eta), \quad \xi_x &= J y_\eta, \quad \xi_y = -J x_\eta \\
\eta_t = J(x_\tau y_\xi - y_\tau x_\xi), \quad \eta_x &= -J y_\xi, \quad \eta_y = J x_\xi
\end{aligned}
$$

Invariants of the Transformation

We would like to avoid having spatially varying coefficients outside of the derivative operators – first multiply by J^{-1}

$$\partial_\tau Q + \xi_t \partial_\xi Q + \eta_t \partial_\eta Q + \xi_x \partial_\xi E + \eta_x \partial_\eta E + \xi_y \partial_\xi F + \eta_y \partial_\eta F = 0$$

$$\left(\frac{\xi_x}{J}\right) \partial_\xi E = \partial_\xi \left(\frac{\xi_x}{J} E\right) - E \partial_\xi \left(\frac{\xi_x}{J}\right)$$

$$\mathrm{Term}_1 + \mathrm{Term}_2 = 0$$

$$\begin{aligned}
\mathrm{Term}_1 &= \partial_\tau (Q/J) + \partial_\xi [(\xi_t Q + \xi_x E + \xi_y F)/J] + \partial_\eta [(\eta_t Q + \eta_x E + \eta_y F)/J] \\
\mathrm{Term}_2 &= -Q[\partial_\tau (J^{-1}) + \partial_\xi (\xi_t/J) + \partial_\eta (\eta_t/J)] \\
&\quad - E[\partial_\xi (\xi_x/J) + \partial_\eta (\eta_x/J)] - F[\partial_\xi (\xi_y/J) + \partial_\eta (\eta_y/J)]
\end{aligned}$$

Invariants of the Transformation

Term_2 includes

$$\begin{aligned}
\partial_\tau(J^{-1}) &+ \partial_\xi(\xi_t/J) + \partial_\eta(\eta_t/J) \\
\partial_\xi(\xi_x/J) &+ \partial_\eta(\eta_x/J) \\
\partial_\xi(\xi_y/J) &+ \partial_\eta(\eta_y/J)
\end{aligned}$$

$$\begin{aligned}
\partial_\tau(x_\xi y_\eta - y_\xi x_\eta) + \partial_\xi(-x_\tau y_\eta + y_\tau x_\eta) &+ \partial_\eta(x_\tau y_\xi - y_\tau x_\xi) \\
\partial_\xi(y_\eta) + \partial_\eta(-y_\xi) \\
\partial_\xi(-x_\eta) + \partial_\eta(x_\xi)
\end{aligned}$$

All equal zero as differentiation is commutative

Navier-Stokes Equations in Generalized Curvilinear Coordinates

$$\partial_\tau \widehat{Q} + \partial_\xi \widehat{E} + \partial_\eta \widehat{F} = Re^{-1}[\partial_\xi \widehat{E}_{\mathrm{v}} + \partial_\eta \widehat{F}_{\mathrm{v}}]$$

$$\widehat{Q} = J^{-1} \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ e \end{bmatrix}$$

$$\widehat{E} = J^{-1} \begin{bmatrix} \rho U \\ \rho u U + \xi_x p \\ \rho v U + \xi_y p \\ U(e+p) - \xi_t p \end{bmatrix}, \ \widehat{F} = J^{-1} \begin{bmatrix} \rho V \\ \rho u V + \eta_x p \\ \rho v V + \eta_y p \\ V(e+p) - \eta_t p \end{bmatrix}$$

$$U = \xi_t + \xi_x u + \xi_y v, \quad V = \eta_t + \eta_x u + \eta_y v$$

Navier-Stokes Equations in Generalized Curvilinear Coordinates

Viscous and heat conduction terms

$$
\begin{aligned}
\tau_{xx} &= \mu(4(\xi_x u_\xi + \eta_x u_\eta) - 2(\xi_y v_\xi + \eta_y v_\eta))/3 \\
\tau_{xy} &= \mu(\xi_y u_\xi + \eta_y u_\eta + \xi_x v_\xi + \eta_x v_\eta) \\
\tau_{yy} &= \mu(-2(\xi_x u_\xi + \eta_x u_\eta) + 4(\xi_y v_\xi + \eta_y v_\eta))/3 \\
f_4 &= u\tau_{xx} + v\tau_{xy} + \mu Pr^{-1}(\gamma - 1)^{-1}(\xi_x \partial_\xi a^2 + \eta_x \partial_\eta a^2) \\
g_4 &= u\tau_{xy} + v\tau_{yy} + \mu Pr^{-1}(\gamma - 1)^{-1}(\xi_y \partial_\xi a^2 + \eta_y \partial_\eta a^2)
\end{aligned}
$$

Navier-Stokes Equations in Generalized Curvilinear Coordinates

Uniform flow preservation is a useful test for a flow solver

$$
\begin{aligned}
\rho &= 1 \\
u &= M_\infty \\
v &= 0 \\
e &= \frac{1}{\gamma(\gamma-1)} + \frac{1}{2}M_\infty^2
\end{aligned}
$$

Thin-Layer Approximation

Thin-Layer Approximation

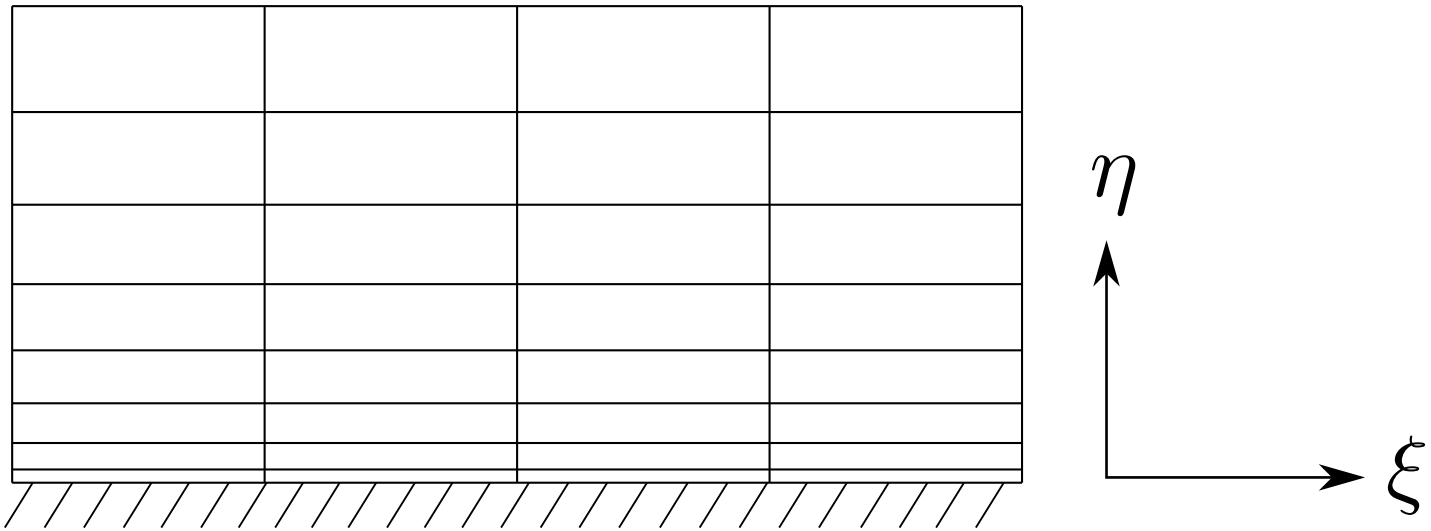
In an attached or mildly separated flow at high Reynolds number, boundary layers are thin

In thin boundary layers, velocity gradients normal to the wall are much higher than those parallel to the wall

Thin-layer approximation neglects streamwise gradients of viscous stresses and heat conduction terms – valid if mesh is body fitted and at least close to orthogonal at the surface

Thin-Layer Approximation

$Re \gg 1$



Mesh near body surface

Thin-Layer Approximation

$$\partial_\tau \widehat{Q} + \partial_\xi \widehat{E} + \partial_\eta \widehat{F} = Re^{-1} \partial_\eta \widehat{S}$$

$$\widehat{S} = J^{-1} \begin{bmatrix} 0 \\ \eta_x m_1 + \eta_y m_2 \\ \eta_x m_2 + \eta_y m_3 \\ \eta_x (u m_1 + v m_2 + m_4) + \eta_y (u m_2 + v m_3 + m_5) \end{bmatrix}$$

Thin-Layer Approximation

$$
\begin{aligned}
m_1 &= \mu(4\eta_x u_\eta - 2\eta_y v_\eta)/3 \\
m_2 &= \mu(\eta_y u_\eta + \eta_x v_\eta) \\
m_3 &= \mu(-2\eta_x u_\eta + 4\eta_y v_\eta)/3 \\
m_4 &= \mu Pr^{-1}(\gamma - 1)^{-1}\eta_x \partial_\eta(a^2) \\
m_5 &= \mu Pr^{-1}(\gamma - 1)^{-1}\eta_y \partial_\eta(a^2)
\end{aligned}
$$

Spatial Differencing

Spatial Differencing

Second-order centered differences for inviscid fluxes and grid metrics (in computational space)

$$\Delta\xi = 1 \qquad \Delta\eta = 1$$

$$Q_{j,k} := Q(j\Delta\xi, k\Delta\eta)$$

$$\delta_\xi \widehat{E}_{j,k} = \frac{\widehat{E}_{j+1,k} - \widehat{E}_{j-1,k}}{2\Delta\xi} \qquad \delta_\eta \widehat{F}_{j,k} = \frac{\widehat{F}_{j,k+1} - \widehat{F}_{j,k-1}}{2\Delta\eta}$$

$$(x_\xi)_{j,k} = \frac{x_{j+1,k} - x_{j-1,k}}{2\Delta\xi}$$

Spatial Differencing

Compact form for second derivatives with spatially varying coefficient

$$\partial_\eta \left(\alpha_{j,k} \partial_\eta \beta_{j,k}\right)$$

$$\begin{aligned}
\left(\frac{\partial \beta}{\partial \eta}\right)_{k+1/2} &= \beta_{j,k+1} - \beta_{j,k} \\
\left(\frac{\partial \beta}{\partial \eta}\right)_{k-1/2} &= \beta_{j,k} - \beta_{j,k-1}
\end{aligned}$$

$$\alpha_{j,k+1/2} = \frac{1}{2}\left(\alpha_{j,k} + \alpha_{j,k+1}\right)$$
$$\alpha_{j,k-1/2} = \frac{1}{2}\left(\alpha_{j,k-1} + \alpha_{j,k}\right)$$

$$\frac{\left(\alpha_{j,k+1} + \alpha_{j,k}\right)}{2}\left(\beta_{j,k+1} - \beta_{j,k}\right) - \frac{\left(\alpha_{j,k} + \alpha_{j,k-1}\right)}{2}\left(\beta_{j,k} - \beta_{j,k-1}\right)$$

Metric Differencing and Invariants

In 2D

$$
\begin{aligned}
\delta_\xi \delta_\eta y_{j,k} - \delta_\eta \delta_\xi y_{j,k} &= \delta_\xi (y_{j,k+1} - y_{j,k-1})/2 - \delta_\eta (y_{j+1,k} - y_{j-1,k})/2 \\
&= [y_{j+1,k+1} - y_{j-1,k+1} - y_{j+1,k-1} + y_{j-1,k-1}]/4 \\
&\quad -[y_{j+1,k+1} - y_{j+1,k-1} - y_{j-1,k+1} + y_{j-1,k-1}]/4 \\
&= 0
\end{aligned}
$$

$$\text{In 3D} \quad \xi_x = J(y_\eta z_\zeta - y_\zeta z_\eta) \quad \text{does not work}$$

$$\text{Can use} \quad \xi_x = J((y_\eta z)_\zeta - (y_\zeta z)_\eta) \quad \text{instead}$$

A Nonlinear Artificial Dissipation Scheme

Purpose

- to eliminate high-frequency modes that are not resolved and can contaminate the solution;
- to enhance stability and convergence to steady state;
- to prevent oscillations at discontinuities, such as shock waves.

A Nonlinear Artificial Dissipation Scheme

Components

- a high-order component for smooth regions of the flow field,
- a first-order component for shock capturing,
- a means of sensing shocks and other discontinuities so that the appropriate dissipation operator can be selected in different regions of the flow field.

A Nonlinear Artificial Dissipation Scheme

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = \frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = 0$$

where $f = Au$

$$\delta_x f = \delta_x^{\mathrm{a}} f + \delta_x^{\mathrm{s}}(|A|u)$$

$$\left(\delta_x^{\mathrm{s}} u\right)_j = \frac{\epsilon_4}{\Delta x}(u_{j-2} - 4u_{j-1} + 6u_j - 4u_{j+1} + u_{j+2}) \propto \epsilon_4 \Delta x^3 \frac{\partial^4 u}{\partial x^4}$$

$$\text{Near shocks}: \ \left(\delta_x^{\mathrm{s}} u\right)_j = \frac{\epsilon_2}{\Delta x}(-u_{j-1} + 2u_j - u_{j+1}) \propto -\epsilon_2 \Delta x \frac{\partial^2 u}{\partial x^2}$$

A Nonlinear Artificial Dissipation Scheme

$$\nabla\Delta\nabla\Delta u_j = u_{j-2} - 4u_{j-1} + 6u_j - 4u_{j+1} + u_{j+2}$$

$$\nabla\Delta u_j = u_{j-1} - 2u_j + u_{j+1}$$

$$D_j = \nabla\Delta\nabla\Delta|A_j|Q_j$$

$$D_j = |A_j|\nabla\Delta\nabla\Delta Q_j \quad \text{nonconservative}$$

$$D_j = \nabla|A_{j+1/2}|\Delta\nabla\Delta Q_j \quad \text{preferred choice}$$

A Nonlinear Artificial Dissipation Scheme

$$\partial_\tau \widehat{Q} = -\delta_\xi \widehat{E} - \delta_\eta \widehat{F} + Re^{-1}[\delta_\xi \widehat{E}_{\mathrm{v}} + \delta_\eta \widehat{F}_{\mathrm{v}}]$$

$$\text{Consider the term}: \ \partial_\tau \widehat{Q} = -\delta_\xi \widehat{E}$$

$$\text{Conservation form}: \ \partial_\tau \widehat{Q} = -(f_{j+1/2} - f_{j-1/2})$$

$$f_{j+1/2} = \frac{1}{2}(\widehat{E}_j + \widehat{E}_{j+1})$$

$$(\partial_\tau \widehat{Q})_{j,k} = -(\delta_\xi \widehat{E})_{j,k} + (D_\xi)_{j,k}$$

A Nonlinear Artificial Dissipation Scheme

$$
\begin{aligned}
(D_\xi)_{j,k} &= \nabla_\xi \left(\epsilon^{(2)}|\widehat{A}|J^{-1}\right)_{j+1/2,k} \Delta_\xi Q_{j,k} \\
&- \nabla_\xi \left(\epsilon^{(4)}|\widehat{A}|J^{-1}\right)_{j+1/2,k} \Delta_\xi \nabla_\xi \Delta_\xi Q_{j,k}
\end{aligned}
$$

$$
\widehat{A} = \frac{\partial \widehat{E}}{\partial \widehat{Q}}
$$

$$
\left(\epsilon^{(2)}|\widehat{A}|J^{-1}\right)_{j+1/2,k} = \frac{1}{2}\left[\left(\epsilon^{(2)}|\widehat{A}|J^{-1}\right)_{j,k} + \left(\epsilon^{(2)}|\widehat{A}|J^{-1}\right)_{j+1,k}\right]
$$

$$
\left(\epsilon^{(4)}|\widehat{A}|J^{-1}\right)_{j+1/2,k} = \frac{1}{2}\left[\left(\epsilon^{(4)}|\widehat{A}|J^{-1}\right)_{j,k} + \left(\epsilon^{(4)}|\widehat{A}|J^{-1}\right)_{j+1,k}\right]
$$

A Nonlinear Artificial Dissipation Scheme

Shock sensor based on pressure:

$$
\begin{aligned}
\epsilon_{j,k}^{(2)} &= \kappa_2 \max(\Upsilon_{j+1,k}, \Upsilon_{j,k}, \Upsilon_{j-1,k}) \\
\Upsilon_{j,k} &= \left| \frac{p_{j+1,k} - 2p_{j,k} + p_{j-1,k}}{p_{j+1,k} + 2p_{j,k} + p_{j-1,k}} \right| \\
\epsilon_{j,k}^{(4)} &= \max(0, \kappa_4 - \epsilon_{j,k}^{(2)})
\end{aligned}
$$

A Nonlinear Artificial Dissipation Scheme

Conservation form

$$(D_\xi)_{j,k} = (d_\xi)_{j+1/2,k} - (d_\xi)_{j-1/2,k}$$

$$\begin{aligned}(d_\xi)_{j+1/2,k} &= \left(\epsilon^{(2)}|\widehat{A}|J^{-1}\right)_{j+1/2,k} \Delta_\xi Q_{j,k} \\ &- \left(\epsilon^{(4)}|\widehat{A}|J^{-1}\right)_{j+1/2,k} \Delta_\xi \nabla_\xi \Delta_\xi Q_{j,k}\end{aligned}$$

Scalar artificial dissipation

$$\sigma = |U| + a\sqrt{\xi_x^2 + \xi_y^2}$$

Implicit Time Marching and the Approximate Factorization Algorithm

Implicit Time Marching and the Approximate Factorization Algorithm

$$\partial_\tau \widehat{Q} = -\delta_\xi \widehat{E} + D_\xi - \delta_\eta \widehat{F} + D_\eta + Re^{-1}[\delta_\xi \widehat{E}_\mathrm{v} + \delta_\eta \widehat{F}_\mathrm{v}]$$

$$\frac{\mathrm{d}\widehat{\mathbf{Q}}}{\mathrm{d}t} = \mathbf{R}(\widehat{\mathbf{Q}})$$

$$R(\widehat{Q}) = -\delta_\xi \widehat{E} + D_\xi - \delta_\eta \widehat{F} + D_\eta + Re^{-1}[\delta_\xi \widehat{E}_\mathrm{v} + \delta_\eta \widehat{F}_\mathrm{v}]$$

Implicit Time Marching and the Approximate Factorization Algorithm

For steady flows:

$$\mathbf{R}(\widehat{\mathbf{Q}}) = \mathbf{0}$$

Newton's method:

$$\mathbf{A}_n \Delta \widehat{\mathbf{Q}}_n = -\mathbf{R}(\widehat{\mathbf{Q}}_n)$$

$$\mathbf{A}_n = \frac{\partial \mathbf{R}}{\partial \widehat{\mathbf{Q}}}$$

$$\Delta \widehat{\mathbf{Q}} = \widehat{\mathbf{Q}}_{n+1} - \widehat{\mathbf{Q}}_n$$

Implicit Time Marching

Two-parameter family

$$\begin{aligned}\widehat{\mathbf{Q}}^{n+1} &= \frac{\theta\Delta t}{1+\varphi}\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\mathbf{Q}}^{n+1} + \frac{(1-\theta)\Delta t}{1+\varphi}\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\mathbf{Q}}^{n} + \frac{1+2\varphi}{1+\varphi}\widehat{\mathbf{Q}}^{n} - \frac{\varphi}{1+\varphi}\widehat{\mathbf{Q}}^{n-1} \\ &\quad + O\left[(\theta - \frac{1}{2} - \varphi)\Delta t^2 + \Delta t^3\right]\end{aligned}$$

$\theta = 1$, $\varphi = 0$ gives implicit Euler, $\theta = 1$, $\varphi = 1/2$ gives 2nd-order backward

Applying implicit Euler to the thin-layer equation gives

$$\widehat{Q}^{n+1} - \widehat{Q}^{n} = h\left(-\delta_\xi \widehat{E}^{n+1} + D_\xi^{n+1} - \delta_\eta \widehat{F}^{n+1} + D_\eta^{n+1} + Re^{-1}\delta_\eta \widehat{S}^{n+1}\right)$$

Local Time Linearization

$$
\begin{aligned}
\widehat{E}^{n+1} &= \widehat{E}^{n} + \widehat{A}^{n}\Delta\widehat{Q}^{n} + O(h^2) \\
\widehat{F}^{n+1} &= \widehat{F}^{n} + \widehat{B}^{n}\Delta\widehat{Q}^{n} + O(h^2) \\
Re^{-1}\widehat{S}^{n+1} &= Re^{-1}\left[\widehat{S}^{n} + \widehat{M}^{n}\Delta\widehat{Q}^{n}\right] + O(h^2)
\end{aligned}
$$

where $\widehat{A} = \partial\widehat{E}/\partial\widehat{Q}$, $\widehat{B} = \partial\widehat{F}/\partial\widehat{Q}$ and $\widehat{M} = \partial\widehat{S}/\partial\widehat{Q}$ are the flux Jacobians, and $\Delta\widehat{Q}^{n}$ is $O(h)$

Local Time Linearization

$$\begin{bmatrix} \kappa_t & \kappa_x & \kappa_y & 0 \\ -u\theta + \kappa_x\phi^2 & \kappa_t + \theta - (\gamma-2)\kappa_x u & \kappa_y u - (\gamma-1)\kappa_x v & (\gamma-1)\kappa_x \\ -v\theta + \kappa_y\phi^2 & \kappa_x v - (\gamma-1)\kappa_y u & \kappa_t + \theta - (\gamma-2)\kappa_y v & (\gamma-1)\kappa_y \\ \theta[\phi^2 - a_1] & \kappa_x a_1 - (\gamma-1)u\theta & \kappa_y a_1 - (\gamma-1)v\theta & \gamma\theta + \kappa_t \end{bmatrix}$$

$$\widehat{a}_{21} = \frac{\partial \widehat{e}_2}{\partial \widehat{q}_1}$$

Local Time Linearization

$$\widehat{Q} = \begin{bmatrix} \widehat{q}_1 \\ \widehat{q}_2 \\ \widehat{q}_3 \\ \widehat{q}_4 \end{bmatrix} = J^{-1} \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ e \end{bmatrix}, \ \widehat{E} = \begin{bmatrix} \widehat{e}_1 \\ \widehat{e}_2 \\ \widehat{e}_3 \\ \widehat{e}_4 \end{bmatrix} = J^{-1} \begin{bmatrix} \rho U \\ \rho u U + \xi_x p \\ \rho v U + \xi_y p \\ U(e+p) - \xi_t p \end{bmatrix}$$

$$\begin{aligned}
\widehat{e}_2 &= J^{-1}\rho u U + J^{-1}\xi_x p \\
&= J^{-1}\rho u \xi_t + J^{-1}\rho u^2 \xi_x + J^{-1}\rho u v \xi_y \\
&\quad + J^{-1}\xi_x(\gamma-1)e - J^{-1}\xi_x(\gamma-1)\frac{1}{2}\rho u^2 - J^{-1}\xi_x(\gamma-1)\frac{1}{2}\rho v^2 \\
&= \xi_t \widehat{q}_2 + \xi_x \frac{\widehat{q}_2^2}{\widehat{q}_1} + \xi_y \frac{\widehat{q}_2 \widehat{q}_3}{\widehat{q}_1} + \xi_x(\gamma-1)\widehat{q}_4 - \frac{\xi_x(\gamma-1)}{2}\frac{\widehat{q}_2^2}{\widehat{q}_1} - \frac{\xi_x(\gamma-1)}{2}\frac{\widehat{q}_3^2}{\widehat{q}_1}
\end{aligned}$$

Local Time Linearization

$$
\begin{aligned}
\widehat{a}_{21} = \frac{\partial \widehat{e}_2}{\partial \widehat{q}_1} &= -\xi_x \frac{\widehat{q}_2^2}{\widehat{q}_1^2} - \xi_y \frac{\widehat{q}_2 \widehat{q}_3}{\widehat{q}_1^2} + \frac{\xi_x(\gamma-1)}{2} \frac{\widehat{q}_2^2}{\widehat{q}_1^2} + \frac{\xi_x(\gamma-1)}{2} \frac{\widehat{q}_3^2}{\widehat{q}_1^2} \\
&= -\xi_x u^2 - \xi_y uv + \frac{\xi_x(\gamma-1)}{2} u^2 + \frac{\xi_x(\gamma-1)}{2} v^2 \\
&= -u(\xi_x u + \xi_y v) + \frac{\xi_x(\gamma-1)}{2}(u^2+v^2)
\end{aligned}
$$

Viscous terms

$$
\widehat{M} = J^{-1} \begin{bmatrix} 0 & 0 & 0 & 0 \\ m_{21} & \alpha_1 \partial_\eta(\rho^{-1}) & \alpha_2 \partial_\eta(\rho^{-1}) & 0 \\ m_{31} & \alpha_2 \partial_\eta(\rho^{-1}) & \alpha_3 \partial_\eta(\rho^{-1}) & 0 \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} J
$$

Local Time Linearization

where

$$
\begin{aligned}
m_{21} &= -\alpha_1 \partial_\eta (u/\rho) - \alpha_2 \partial_\eta (v/\rho) \\
m_{31} &= -\alpha_2 \partial_\eta (u/\rho) - \alpha_3 \partial_\eta (v/\rho) \\
m_{41} &= \alpha_4 \partial_\eta \left[-(e/\rho^2) + (u^2 + v^2)/\rho \right] \\
&\quad -\alpha_1 \partial_\eta (u^2/\rho) - 2\alpha_2 \partial_\eta (uv/\rho) \\
&\quad -\alpha_3 \partial_\eta (v^2/\rho) \\
m_{42} &= -\alpha_4 \partial_\eta (u/\rho) - m_{21} \\
m_{43} &= -\alpha_4 \partial_\eta (v/\rho) - m_{31} \\
m_{44} &= \alpha_4 \partial_\eta (\rho^{-1}) \\
\alpha_1 &= \mu[(4/3){\eta_x}^2 + {\eta_y}^2], \quad \alpha_2 = (\mu/3)\eta_x \eta_y \\
\alpha_3 &= \mu[{\eta_x}^2 + (4/3){\eta_y}^2], \quad \alpha_4 = \gamma \mu Pr^{-1}({\eta_x}^2 + {\eta_y}^2)
\end{aligned}
$$

Local Time Linearization

Let's derive the m_{22} term

$$\begin{aligned}\widehat{s}_2 &= \frac{\alpha_1}{J} u_\eta + \frac{\alpha_2}{J} v_\eta \\ &= \frac{\alpha_1}{J} \frac{\partial}{\partial \eta} \left(\frac{\widehat{q}_2}{\widehat{q}_1} \right) + \frac{\alpha_2}{J} \frac{\partial}{\partial \eta} \left(\frac{\widehat{q}_3}{\widehat{q}_1} \right)\end{aligned}$$

$$f(\widehat{q}_2) = \frac{\alpha_1}{J} \frac{\partial}{\partial \eta} \left(\frac{\widehat{q}_2}{\widehat{q}_1} \right)$$

Local Time Linearization

$$
\begin{aligned}
\frac{\partial f}{\partial \widehat{q}_2}\Delta\widehat{q}_2 &= \lim_{\epsilon\to 0}\frac{f(\widehat{q}_2+\epsilon\Delta\widehat{q}_2)-f(\widehat{q}_2)}{\epsilon} \\
&= \lim_{\epsilon\to 0}\left[\frac{\alpha_1}{J}\frac{\partial}{\partial\eta}\left(\frac{\widehat{q}_2+\epsilon\Delta\widehat{q}_2}{\widehat{q}_1}\right)-\frac{\alpha_1}{J}\frac{\partial}{\partial\eta}\left(\frac{\widehat{q}_2}{\widehat{q}_1}\right)\right]/\epsilon \\
&= \lim_{\epsilon\to 0}\left[\frac{\alpha_1}{J}\frac{\partial}{\partial\eta}\left(\frac{\epsilon\Delta\widehat{q}_2}{\widehat{q}_1}\right)\right]/\epsilon \\
&= \frac{\alpha_1}{J}\frac{\partial}{\partial\eta}\left(\frac{\Delta\widehat{q}_2}{\widehat{q}_1}\right)
\end{aligned}
$$

$$
\widehat{m}_{22}\Delta\widehat{q}_2 = J^{-1}\alpha_1\frac{\partial}{\partial\eta}\left(\frac{J}{\rho}\Delta\widehat{q}_2\right)
$$

Local Time Linearization

$$\left[I + h\delta_\xi \widehat{A}^n - hL_\xi \quad + \quad h\delta_\eta \widehat{B}^n - hL_\eta - Re^{-1} h\, \delta_\eta \widehat{M}\right] \Delta\widehat{Q}^n =$$
$$- \quad h\left(\delta_\xi \widehat{E}^n - D_\xi^n + \delta_\eta \widehat{F}^n - D_\eta^n - Re^{-1}\delta_\eta \widehat{S}^n\right)$$

Difference operators operate on the product, including $\Delta\widehat{Q}^n$, for example

$$\frac{1}{2}h(\widehat{A}^n_{j+1,k}\Delta\widehat{Q}^n_{j+1,k} - \widehat{A}^n_{j-1,k}\Delta\widehat{Q}^n_{j-1,k})$$

Keep in mind that $\Delta\widehat{Q}^n$ is unknown – we are solving for it

Matrix Form of the Unfactored Algorithm

$$\left[I + h\delta_\xi \widehat{A}^n + h\delta_\eta \widehat{B}^n\right] \Rightarrow$$

$$\begin{bmatrix}
I & h\widehat{A}/2 & & & h\widehat{B}/2 & & & & & & \\
-h\widehat{A}/2 & I & h\widehat{A}/2 & & & h\widehat{B}/2 & & & & & \\
 & -h\widehat{A}/2 & I & h\widehat{A}/2 & & & h\widehat{B}/2 & & & & \\
 & & -h\widehat{A}/2 & I & h\widehat{A}/2 & & & h\widehat{B}/2 & & & \\
-h\widehat{B}/2 & & & -h\widehat{A}/2 & I & h\widehat{A}/2 & & & h\widehat{B}/2 & & \\
 & \ddots & & & \ddots & \ddots & \ddots & & & \ddots & \\
 & & -h\widehat{B}/2 & & & -h\widehat{A}/2 & I & h\widehat{A}/2 & & & h\widehat{B}/2 \\
 & & & -h\widehat{B}/2 & & & -h\widehat{A}/2 & I & h\widehat{A}/2 & & \\
 & & & & -h\widehat{B}/2 & & & -h\widehat{A}/2 & I & h\widehat{A}/2 & \\
 & & & & & \ddots & & & \ddots & \ddots & \ddots \\
 & & & & & & -h\widehat{B}/2 & & & -h\widehat{A}/2 & I & h\widehat{A}/2 \\
 & & & & & & & -h\widehat{B}/2 & & & -h\widehat{A}/2 & I
\end{bmatrix}$$

Approximate Factorization

Left-hand side terms:

$$\left[I + h\delta_\xi \widehat{A}^n + h\delta_\eta \widehat{B}^n - hRe^{-1}\delta_\eta \widehat{M}^n\right] \Delta\widehat{Q}^n =$$
$$\left[I + h\delta_\xi \widehat{A}^n\right] \left[I + h\delta_\eta \widehat{B}^n - hRe^{-1}\delta_\eta \widehat{M}^n\right] \Delta\widehat{Q}^n$$
$$-h^2\delta_\xi \widehat{A}^n \delta_\eta \widehat{B}^n \Delta\widehat{Q}^n + h^2 Re^{-1}\delta_\xi \widehat{A}^n \delta_\eta \widehat{M}^n \Delta\widehat{Q}^n$$

Neglect $O(h^3)$ terms to obtain

$$\begin{aligned}\left[I + h\delta_\xi \widehat{A}^n\right] \left[I + h\delta_\eta \widehat{B}^n \right. &- \left. hRe^{-1}\delta_\eta \widehat{M}^n\right] \Delta\widehat{Q}^n = \\ &- h\left[\delta_\xi \widehat{E}^n + \delta_\eta \widehat{F}^n - Re^{-1}\delta_\eta \widehat{S}^n\right]\end{aligned}$$

Approximate Factorization

$$\left[I + h\delta_\xi \widehat{A}^n\right] \Rightarrow \begin{bmatrix} I & h\widehat{A}/2 & & & & & & \\ -h\widehat{A}/2 & I & h\widehat{A}/2 & & & & & \\ & -h\widehat{A}/2 & I & h\widehat{A}/2 & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & -h\widehat{A}/2 & I & h\widehat{A}/2 & & \\ & & & & -h\widehat{A}/2 & I & h\widehat{A}/2 & \\ & & & & & -h\widehat{A}/2 & I & h\widehat{A}/2 \\ & & & & & & -h\widehat{A}/2 & I \end{bmatrix}$$

$$\left[I + h\delta_\xi \widehat{A}^n\right] \Delta\tilde{Q} = -h\left[\delta_\xi \widehat{E}^n + \delta_\eta \widehat{F}^n - Re^{-1}\delta_\eta \widehat{S}^n\right]$$

$$\left[I + h\delta_\eta \widehat{B}^n - hRe^{-1}\delta_\eta \widehat{M}^n\right] \Delta\widehat{Q}^n = \Delta\tilde{Q}$$

Diagonal Form of The Implicit Algorithm

Matrices $\widehat{A}$ and $\widehat{B}$ can be diagonalized:

$$\Lambda_\xi = T_\xi^{-1}\widehat{A}T_\xi \quad \text{and} \quad \Lambda_\eta = T_\eta^{-1}\widehat{B}T_\eta$$

Without approximation we can write for the Euler equations

$$\left[T_\xi\, T_\xi^{-1} + h\,\delta_\xi\left(T_\xi\,\Lambda_\xi\,T_\xi^{-1}\right)\right]\,\left[T_\eta\, T_\eta^{-1} + h\,\delta_\eta\left(T_\eta\,\Lambda_\eta\,T_\eta^{-1}\right)\right]\,\Delta\widehat{Q}^n$$
$$= -h\left[\delta_\xi\widehat{E}^n + \delta_\eta\widehat{F}^n\right] = \widehat{R}^n$$

Pull eigenvector matrices out of spatial operator

$$T_\xi\left[I + h\,\delta_\xi\,\Lambda_\xi\right]\,\widehat{N}\,\left[I + h\,\delta_\eta\,\Lambda_\eta\right]\,T_\eta^{-1}\Delta\widehat{Q}^n = \widehat{R}^n$$

where $\widehat{N} = T_\xi^{-1}T_\eta$

Local Time Stepping

Goal is to maintain roughly constant (optimal) Courant number

$$C_{\mathrm{n}} = \frac{(|u| + a)h}{\Delta x}$$

$$\Delta t = \frac{\Delta t_{\mathrm{ref}}}{|U| + |V| + a\sqrt{\xi_x^2 + \xi_y^2 + \eta_x^2 + \eta_y^2}}$$

Variation in mesh spacing typically far exceeds variation in wave speed

$$\Delta t = \frac{\Delta t|_{\mathrm{ref}}}{1 + \sqrt{J}}$$

Dual Time Stepping for Unsteady Flow Computations

Consider second-order backward implicit time-marching:

$$u_{n+1} = \frac{1}{3}[4u_n - u_{n-1} + 2hu'_{n+1}]$$

Gives

$$\begin{aligned}
\widehat{Q}^{n+1} &= \frac{4}{3}\widehat{Q}^n - \frac{1}{3}\widehat{Q}^{n-1} \\
&+ \frac{2h}{3}\left(-\delta_\xi \widehat{E}^{n+1} + D_\xi^{n+1} - \delta_\eta \widehat{F}^{n+1} + D_\eta^{n+1} + Re^{-1}\delta_\eta \widehat{S}^{n+1}\right)
\end{aligned}$$

Dual Time Stepping for Unsteady Flow Computations

After local time linearization and approximate factorization we obtain

$$\left[I + \frac{2h}{3}\delta_\xi \widehat{A}^n\right] \qquad \left[I + \frac{2h}{3}\delta_\eta \widehat{B}^n - \frac{2h}{3}Re^{-1}\delta_\eta \widehat{M}^n\right]\Delta\widehat{Q}^n =$$

$$\widehat{Q}^n - \widehat{Q}^{n-1} - \frac{2h}{3}\left[\delta_\xi \widehat{E}^n + \delta_\eta \widehat{F}^n - Re^{-1}\delta_\eta \widehat{S}^n\right]$$

Rather than linearizing in time, we can solve this nonlinear problem at each time step to eliminate linearization and factorization error

Dual Time Stepping for Unsteady Flow Computations

$$\frac{3\widehat{Q}^{n+1} - 4\widehat{Q}^{n} + \widehat{Q}^{n-1}}{2h} + R(\widehat{Q}^{n+1}) = 0$$

$$R(\widehat{Q}^{n+1}) = \left[\delta_\xi \widehat{E}^{n+1} - D_\xi^{n+1} + \delta_\eta \widehat{F}^{n+1} - D_\eta^{n+1} - Re^{-1}\delta_\eta \widehat{S}^{n+1}\right]$$

$$R_{\mathrm{u}}(\widehat{Q}) = \frac{3\widehat{Q} - 4\widehat{Q}^{n} + \widehat{Q}^{n-1}}{2h} + R(\widehat{Q})$$

$$R_{\mathrm{u}}(\widehat{Q}) = 0$$

Dual Time Stepping for Unsteady Flow Computations

$$\frac{\mathrm{d}\widehat{Q}}{\mathrm{d}\tau} + R_{\mathrm{u}}(\widehat{Q}) = 0$$

$$\left[I + \frac{\Delta\tau}{b}\delta_\xi \widehat{A}^p\right]\left[I + \frac{\Delta\tau}{b}\delta_\eta \widehat{B}^p \quad - \quad \frac{\Delta\tau}{b} Re^{-1}\delta_\eta \widehat{M}^p\right]\Delta\widehat{Q}^p = \\ - \quad \frac{\Delta\tau}{b} R_{\mathrm{u}}(\widehat{Q}^p)$$

$$b = 1 + \frac{3\Delta\tau}{2h}$$

$\Delta\tau$ can be chosen for fast convergence (independent of Δt, which is chosen for time accuracy)

Boundary Conditions

Characteristic Approach (Euler Equations)

Consider 1D Euler equations

$$\partial_t Q + \partial_x (AQ) = 0$$

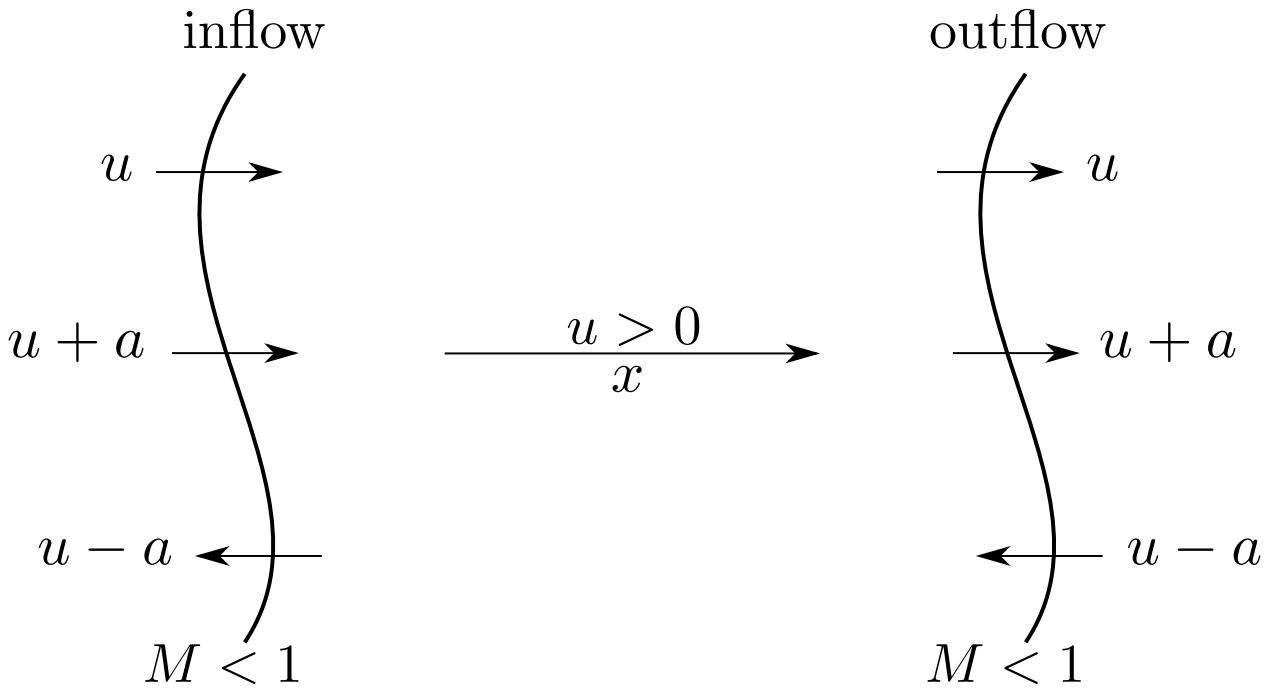
Eigenvalues of A are

$$\Lambda_A = \begin{bmatrix} u & 0 & 0 \\ 0 & u+a & 0 \\ 0 & 0 & u-a \end{bmatrix}$$

Diagonalized form

$$\partial_t \left(X^{-1}Q\right) + \Lambda_A \partial_x \left(X^{-1}Q\right) = 0$$

Characteristic Approach (Euler Equations)



Characteristics at subsonic inflow and outflow boundaries of a closed domain

In a supersonic flow all eigenvalues have the same sign

Boundary Conditions for External Flows

Body Surfaces

Inviscid flow: flow tangency condition

$$V_n = \frac{\eta_x u + \eta_y v}{\sqrt{\eta_x^2 + \eta_y^2}}, \quad V_t = \frac{\eta_y u - \eta_x v}{\sqrt{\eta_x^2 + \eta_y^2}}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{\sqrt{\eta_x^2 + \eta_y^2}} \begin{bmatrix} \eta_y & \eta_x \\ -\eta_x & \eta_y \end{bmatrix} \begin{pmatrix} V_t \\ V_n \end{pmatrix}$$

Set V_n to zero, extrapolate V_t from interior

Solve for values of u and v at the surface

Boundary Conditions for External Flows

Body Surfaces

Viscous flow: no-slip condition plus adiabatic or isothermal condition:

$$u = v = 0 \quad \text{and} \quad \frac{\partial T}{\partial n} = 0 \quad \text{or specify } T$$

Boundary Conditions for External Flows

Far-Field Boundaries

$$\begin{aligned}
\lambda_1 &= V_n - a, & R_1 &= V_n - 2a/(\gamma - 1) \\
\lambda_2 &= V_n + a, & R_2 &= V_n + 2a/(\gamma - 1) \\
\lambda_3 &= V_n, & R_3 &= S = \ln \frac{p}{\rho^\gamma} \quad \text{(entropy)} \\
\lambda_4 &= V_n, & R_4 &= V_t
\end{aligned}$$

$$\text{Subsonic inflow } (V_n \leq 0): \quad \lambda_1 < 0,\ \lambda_2 > 0,\ \lambda_3 < 0,\ \lambda_4 < 0$$

$\rightarrow$ Set R_1, R_3, and R_4, extrapolate R_2 from interior

$$\text{Subsonic inflow } (V_n \leq 0): \quad \lambda_1 < 0,\ \lambda_2 > 0,\ \lambda_3 > 0,\ \lambda_4 > 0$$

$\rightarrow$ Set R_1, extrapolate R_2, R_3, and R_4 from interior